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TORONTO

APPLICATIONS OF THE ABSOLUTE DIFFERENTIAL CALCULUS

BY

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PREFACE.

The absolute differential calculus came into prominence as the instrument best fitted for dealing with the general theory of relativity and it has also been found indispensable for the differential geometry of hyperspaces. Although many books have appeared dealing with this calculus in relation to these advanced théories, there are very few that use it to develop more elementary subjects. The present book was written with the object of providing such a text-book, enabling the student to become acquainted with tensor methods much earlier in his mathematical education. There is no better way in which the student can appreciate the power of tensorial methods than by seeing them applied to subjects with which he is already familiar, and the topics treated here do not go outside the ordinary university honours course in mathematics. Naturally the book is not intended to give a full account of the subjects chosen, but the author has tried to give a brief consistent treatment of each.

The contents are divided into four parts. The first has been included in order to make the book independent of other works and contains an elementary exposition of the fundamental ideas and notation of tensor theory. The second part gives the application of the calculus to algebraic geometry and is really the geometrical treatment of tensor algebra. Thus the first half of the book deals with the algebraic, as distinct from the differential, properties of tensors and only linear transformations are considered.

The third part introduces the absolute differential calculus proper, namely, the theory of the differentiation of tensors. It deals with elementary differential geometry and approaches the problem of covariant differentiation from the geometrical point of view. It is hoped that this section will be of value to the student as an introduction to advanced differential geometry. The method of treatment has been chosen with this object in view.

The fourth section treats of applied mathematics, giving applications to dynamics, electricity, elasticity and hydrodynamics. A short account has been given of the geometrisation of general dynamics. The final chapter deals with the special theory of relativity from the tensor point of view, in the hope that it will prove a useful introduction to the more difficult general theory. An appendix has been added on the use of orthogonal curvilinear coordinates in mathematical physics, in which the notation of the text is connected with that commonly used in text-books that do not employ tensor notation.

The absence of examples in most of the books on the tensor calculus that have appeared up to the present is a serious disadvantage. In the present work there is a large collection of examples and it is hoped that they will enable the reader to acquire facility in working out problems by tensor methods. Answers to the examples are usually given and in many cases hints for solution are added.

My best thanks are due to Professor T. Levi-Civita, Professor A. Palatini, Professor J. L. Synge and Dr. John Dougall for valuable suggestions during the preparation of the manuscript.

A. J. McCONNELL.

Trinity College, Dublin,
April 1931.

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PART I

Algebraic Preliminaries

CHAPTER I

NOTATION AND DEFINITIONS

1. The indicial notation.

The notation of the absolute differential calculus or, as it is frequently called, the tensor calculus, is so much an integral part of the calculus that once the student has become accustomed to its peculiarities he will have gone a long way towards solving the difficulties of the theory itself. We shall therefore devote the present chapter to a discussion of the notation alone, applying it briefly to the theory of determinants, and we shall postpone to the next chapter the tensor theory proper.

If we are given a set of *three* independent variables, they may be denoted by three different letters, such as x, y, z , but we shall find it more convenient to denote the variables by the same letter, distinguishing them by means of indices. Thus we may write the three variables x_1, x_2, x_3 , or, as they may be more compactly written,

$$x_r \quad (r = 1, 2, 3). \dots \dots \dots (1)$$

Now in (1) we have written the index r as a subscript, but we could equally well have used superscripts instead, so that the variables would be written x^1, x^2, x^3 , or

$$x^r \quad (r = 1, 2, 3). \dots \dots \dots (2)$$

Here it must be understood that x^r does not mean that x is raised to the r th power, but r is used merely to distinguish the three variables. In the sequel we shall have occasion to use both subscripts and superscripts, and in the next chapter we shall give a special significance to the position of the

index. In fact we shall find that, in accordance with the convention adopted later, the form (2) is the appropriate one for our variables and not (1).

A homogeneous linear function of the variables is obviously of the form

$$\sum_{m=1}^3 a_m x^m \equiv a_1 x^1 + a_2 x^2 + a_3 x^3, \quad (3)$$

where a_1, a_2, a_3 are constants. Thus the coefficients of a linear form can be written

$$a_r \quad (r = 1, 2, 3).$$

Systems of quantities, which, like x^r and a_r , depend on one index only, are called *systems of the first order* or *simple systems*, and the separate terms, x^1, x^2, x^3 and a_1, a_2, a_3 , are called the *elements* or *components* of the system. It is obvious that a system of the first order has three components. Moreover there are two types of simple systems, namely, those in which the index is above and those in which it is below, and hence all simple systems are of one or other of the two forms

$$x^r, a_r \quad (r = 1, 2, 3). \quad (4)$$

On the other hand, a homogeneous quadratic function of the variables is of the form

$$\begin{aligned} \sum_{m,n=1}^3 a_{mn} x^m x^n \equiv & a_{11} (x^1)^2 + a_{12} x^1 x^2 + a_{13} x^1 x^3 \\ & + a_{21} x^2 x^1 + a_{22} (x^2)^2 + a_{23} x^2 x^3 \\ & + a_{31} x^3 x^1 + a_{32} x^3 x^2 + a_{33} (x^3)^2 \end{aligned} \quad (5)$$

the a 's being again constants. We see that the coefficients of a quadratic form depend on *two* indices, and are written

$$a_{mn} \quad (m, n = 1, 2, 3).$$

Systems, which depend on two indices, are called *double systems* or *systems of the second order*. Since the indices may be either above or below, it follows that double systems can be of three types

$$a_{rs}, a_s^r, a^{rs} \quad (r, s = 1, 2, 3), \quad (6)$$

and it is easily seen that there are 9 components in each system.

Similarly we have *systems of the third order*, which depend on three indices and which can be of four types,

$$a_{rst}, a_{st}^r, a_t^{rs}, a^{rst} \quad (r, s, t = 1, 2, 3). \quad . . . (7)$$

Here each system contains 3^3 or 27 components. We can continue in this way to systems of any order.

To complete the series we shall call a single quantity, a , which has no indices, a *system of order zero*.

Although we have taken the number of variables to be three for definiteness, all that has been said applies also to any number of variables except that the range of values for the indices will be different. For example, if the number of variables were four, the indices would range from 1 to 4 instead of from 1 to 3 as above.

EXAMPLES.

1. Show that a system of the fourth order may be of any one of five types.

2. If the number of variables were four, how many components would there be in systems of the second and third orders?

2. The summation convention.

We shall now adopt two important conventions with regard to indices. In the tensor calculus we very frequently deal with sums of the types (3) and (5), and it is to be noted that in these formulae the indices to which the Σ sign refers appear *twice* in every case. It will add very much to the compactness of our formulae if we can dispense with the Σ sign, and this can be effected if we agree that a Σ sign is understood whenever an index is *repeated* in a single term. Thus (3) would now read

$$a_m x^m \equiv a_1 x^1 + a_2 x^2 + a_3 x^3, \quad (8)$$

and (5) becomes

$$a_{mn} x^m x^n \equiv \left. \begin{aligned} & a_{11} (x^1)^2 + a_{12} x^1 x^2 + a_{13} x^1 x^3 \\ & + a_{21} x^2 x^1 + a_{22} (x^2)^2 + a_{23} x^2 x^3 \\ & + a_{31} x^3 x^1 + a_{32} x^3 x^2 + a_{33} (x^3)^2 \end{aligned} \right\} . . (9)$$

The only inconvenience possessed by our convention arises when we wish to write down the general term in either of the sums (8) or (9). It is very seldom that we require to do this, but we can provide for such an eventuality by agreeing that the summation convention will only apply when the repeated index is a *small italic letter* and by using capital letters for repeated indices which are *not* to be summed. Thus the general terms in the sums (8) and (9) will be denoted by

$$a_M x^M, \quad a_{MN} x^M x^N,$$

respectively.

Our first convention therefore reads:

A repeated small italic index is to be summed from 1 to 3.

Since the repeated index is to be summed from 1 to 3, it follows that the particular letter used for the repeated index is quite immaterial, and we may use for it any letter we please without altering the meaning of the expression we are dealing with. Thus

$$a_m x^m \equiv a_r x^r \equiv a_1 x^1 + a_2 x^2 + a_3 x^3,$$

and

$$a_{mn} x^m x^n \equiv a_{rs} x^r x^s.$$

For this reason the repeated index is often referred to as a *dummy index*.

An index which is not repeated in any single term is known as a *free index*. Thus all the indices in the formulae (4), (6) and (7) are free indices, and it is to be noted that in these formulae the free indices have the range of values from 1 to 3. We shall hence take for our second convention:

A free or unrepeated small italic index is to have the range of values 1, 2, 3.

For example, a system of the second order will now read

$$a_{rs}$$

without any explicit mention of the range of values of r and s . In other words, a_{rs} stands for any one of the 9 quantities

$$\begin{array}{ccc} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33}. \end{array}$$

Although the fact has no significance for the present, we remark that we shall find almost always that a dummy index will appear once as a subscript and once as a superscript, and we shall as far as possible keep to such an arrangement in this chapter.

EXAMPLES.

1. Write out in full the system of linear equations given by

$$a_{rs} x^s = b_r.$$

2. How many terms are included in the sum $a_{mnp} x^m y^n z^p$?
3. Show that a_s^{rs} is a system of the first order, and write out its components in full.

3. Addition, multiplication, and contraction of systems.

In dealing with the algebra of systems there are three main algebraical operations, which are called addition, multiplication and contraction.

(i) *Addition*. This operation applies only to systems of the *same* order and type. If we are given two systems of the same order and type, and if we add each component of the first system to the corresponding component of the second, we obviously arrive at a system of the same order and type as the original systems. This process is the operation of *addition* and the resulting system is called the *sum* of the two systems. Thus, if a_{st}^r and b_{st}^r are two triple systems, then the system c_{st}^r , defined by the equations

$$c_{st}^r = a_{st}^r + b_{st}^r, \dots \dots \dots (10)$$

is the sum of a_{st}^r and b_{st}^r . We are here speaking of the algebraic sum so that subtraction is included as a particular case. Moreover the operation can be immediately extended to find the sum of any number of systems, provided they are all of the same order and type.

(ii) *Multiplication*. We shall now define the *product* of two systems. If we take two systems of any kind and multiply each component of the first by each component of the second, we get a system whose order equals the sum of the orders of the two original systems, and this resulting system is called the *product* of these systems. For example, if a_{st}^r is a triple system and b^{mn} is a double system, we see that the system c_{st}^{rmn} , whose components are given by the equations

$$c_{st}^{rmn} = a_{st}^r b^{mn}, \dots \dots \dots (11)$$

is a system of the fifth order and is the product of a_{st}^r and b^{mn} . The process can of course be extended to define the product of any number of systems.

(iii) *Contraction*. The process of contraction can best be explained by an example. Let us take a system of the fifth order,

$$a_{stu}^{rp}$$

which has both upper and lower indices. If we now put u equal to p , we get the system a_{stp}^p , and since p is now a repeated index

it must be summed from 1 to 3 in accordance with our convention. Thus the new system, obtained in this way, is

$$a_{stp} \equiv \sum_{p=1}^3 a_{stp}^{rp} \equiv a_{st1}^{r1} + a_{st2}^{r2} + a_s^r \quad (12)$$

We therefore see that our new system (12) is of the *third* order, that is, its order is two less than that of the original system. The operation can evidently be repeated several times for we can contract with respect to any pair of indices, one of which is a subscript and the other a superscript. In the example given above we can contract once more with respect to r and t , obtaining the simple system,

$$a_{srp}^{rp} \equiv \sum_{r,p=1}^3 a_{srp}^{rp}.$$

There is another operation on systems, called *composition* or *inner multiplication*, which is not a new operation but is really a combination of multiplication and contraction. To compound two systems we first take their product and then contract this product with respect to a subscript of one and a superscript of the other. Thus an inner product of two systems, a_s^r and b^{mn} , is the system

$$c_s^{rn} = a_s^r b^{mn} = \sum_{p=1}^3 a_{sp}^r b^{mp}.$$

4. Symmetric and skew-symmetric systems.

If we are given a system, a_{mn} , with two subscripts, it may happen that each component is unaltered in value when the indices are interchanged, that is,

$$a_{mn} = a_{nm}.$$

The system is then said to be *symmetric*. More generally, a system having any number of subscripts is said to be symmetric in two of these subscripts if the components of the system are unaltered when the two indices are interchanged, and the system is *completely* symmetric in the subscripts if the interchange of any two subscripts leaves the components unaltered. A completely symmetric system of the third order will therefore satisfy the relations

$$a_{mnp} = a_{mnp} = a_{nmp} = a_{npm} = a_{pmn} = a_{pnm}.$$

On the other hand, the system a_{mn} is *skew-symmetric* if the

interchange of indices alters the sign of the component but not its numerical value, in which case we have

$$a_{mn} = -a_{nm}.$$

These equations, written in full, read

$$\begin{aligned} a_{11} &= -a_{11}, & a_{22} &= -a_{22}, & a_{33} &= -a_{33}, \\ a_{12} &= -a_{21}, & a_{23} &= -a_{32}, & a_{31} &= -a_{13}, \end{aligned}$$

from which we deduce immediately that $a_{11} = a_{22} = a_{33} = 0$. Similarly a system may be skew-symmetric with respect to two subscripts, or may be completely skew-symmetric in all its subscripts. A completely skew-symmetric system of the third order will satisfy the relations

$$a_{mnp} = -a_{mnp} = -a_{nmp} = a_{pnm} = -a_{pnm}.$$

The above remarks about symmetry and skew-symmetry apply equally well to superscripts.

EXAMPLES.

1. How many distinct components are there in the completely symmetric system of the third order?
2. Show that the skew-symmetric system of the third order has only one non-zero distinct component.
3. If a_{mn} is skew-symmetric, prove that

$$a_{mn} x^m x^n = 0,$$

and, conversely, prove that if this equation is true for all values of the variables x^r then a_{mn} is skew-symmetric.

4. If a_{rs} is a double system satisfying the equations

$$ba_{rs} + ca_{sr} = 0,$$

prove that either $b = -c$ and a_{rs} is symmetric, or $b = c$ and a_{rs} is skew-symmetric.

[The indices r and s can have any of the values 1, 2, 3 and hence the above set of equations can be written in the equivalent form $ba_{sr} + ca_{rs} = 0$. Adding to the above equation we get $(b+c)(a_{rs} + a_{sr}) = 0$. Therefore either $b = -c$ or $a_{rs} = -a_{sr}$, etc...]

5. The skew-symmetric three-systems and the Kronecker deltas.

Let a_{rst} be a completely skew-symmetric system of the third order, that is, its components are altered in sign but not in absolute value when any two of its indices are interchanged. Conse-

quently we see that its components can have only the following three distinct values:

- (i) 0, when any two of the indices are equal,
- (ii) $+a_{123}$, when rst is an *even* permutation of 123,
- (iii) $-a_{123}$, when rst is an *odd* permutation of 123.

We shall denote by e_{rst} the skew-symmetric system whose components have the values 0, $+1$, -1 , and we shall refer to it as the *e-system*. We have $e_{123} = +1$, and, if a_{rst} is any skew-symmetric system of the third order, then

$$a_{rst} = a_{123} e_{rst} \dots \dots \dots (13)$$

We can also have all the indices superscripts, so that we have the corresponding skew-symmetric system e^{rst} , whose elements have exactly the same values as those of e_{rst} . We shall find that both these systems are of great value in dealing with determinants.

Now from these two systems, e_{rst} and e^{rst} , we can deduce other systems by the operations of multiplication and contraction. By multiplication we have the system of the sixth order,

$$\delta_{mnp}^{rst} = e^{rst} e_{mnp} \dots \dots \dots (14)$$

Having regard to the values which the *e*-systems can take, we see that the new system has the following values:

- (i) 0, when two or more of the subscripts (or superscripts) are the same,
- (ii) $+1$, when rst and mnp differ by an *even* number of permutations,
- (iii) -1 , when rst and mnp differ by an *odd* number of permutations.

Let us next contract δ_{mnp}^{rst} , thus giving us the system of the fourth order

$$\delta_{mn}^{rs} = \delta_{mnp}^{rsp} = \delta_{mn1}^{rs1} + \delta_{mn2}^{rs2} + \delta_{mn3}^{rs3} \dots \dots (15)$$

We first observe that this is zero if r and s are the same or if m and n are the same. Also if we give r, s the special values 1, 2 respectively, we see from (15) that

$$\delta_{mn}^{12} = \delta_{mn3}^{123},$$

and therefore δ_{mn}^{12} vanishes unless m, n is a permutation of 1, 2, when it is +1 for an even permutation and -1 for an odd permutation. We have similar results if we give r, s any other special values. Hence δ_{mn}^{rs} takes the values:

- (i) 0, when two of the subscripts or superscripts are the same, or when the superscripts have not the same two values as the subscripts,
- (ii) +1, when r, s and m, n are the same permutation of the same two numbers,
- (iii) -1, when r, s and m, n are opposite permutations of the same two numbers.

If we contract δ_{mn}^{rs} , and halve the result, we obtain the double system,

$$\sigma_m^r = \frac{1}{2} \sigma_{mn}^{rn} = \frac{1}{2} (\sigma_{m1}^{r1} + \sigma_{m2}^{r2} + \sigma_{m3}^{r3}). \quad \dots (16)$$

When we put $r = 1$ in δ_m^r , we get

$$\delta_m^1 = \frac{1}{2} (\delta_{m2}^{12} + \delta_{m3}^{13}),$$

and we see that δ_m^1 vanishes unless $m = 1$, in which case it equals +1. The results are similar when we put r equal to 2 or 3. Hence δ_m^r has the values

- (i) 0, if r and m are not equal,
- (ii) +1, if r equals m .

All these δ -systems are generally referred to as the *Kronecker deltas*.

EXAMPLES.

1. Prove that

$$\delta_r^r = 3, \quad \delta_{rst}^{rst} = 2\delta_m^r, \quad \delta_{rst}^{rst} = 3!$$

2. Prove that

$$\delta_{tnt}^{rst} = \delta_{ntn}^{rst} = \delta_{ntn}^{rst}.$$

3. Prove the formulae:

$$\delta_a^r a^s = a^r, \quad \delta_{mn}^{rs} a^{mn} = a^{rs} - a^{sr},$$

$$\delta_{mnp}^{rst} a^{mnp} = a^{rst} - a^{rts} + a^{str} - a^{srt} + a^{trp} - a^{trp}.$$

[To prove the second result, let r, s take the special values R, S . We know that δ_{mn}^{RS} vanishes except when m, n take the values R, S , and we have

$$\delta_{RS}^{RS} = -\delta_{SR}^{RS} = 1.$$

Consequently (remembering our agreement in § 2 as to capital indices),

$$\delta_{mn}^{RS} a^{mn} = \delta_{RS}^{RS} a^{RS} + \delta_{RS}^{RS} a^{SR} = a^{RS} - a^{SR},$$

which proves the result. The others follow similarly.]

DETERMINANTS

6. The determinant formed by a double system α_s^r .

We shall now show how our notation may be employed to develop the elements of the theory of determinants. The determinant of the third order, whose elements are the double system α_s^r , is written

$$|\alpha_s^r| \equiv \begin{vmatrix} \alpha_1^1 & \alpha_2^1 & \alpha_3^1 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 \end{vmatrix}. \quad (17)$$

Here the upper index denotes the row and the lower index the column. If the determinant is expanded in full by columns, it reads, by the definition of the determinant,

$$\sum_{i,j,k}^3 \pm a_1^i a_2^j a_3^k,$$

where i, j, k is a permutation of 1, 2, 3, and the plus or minus sign is given according as the permutation is even or odd. Remembering the properties of the e -systems, we therefore see that the determinant equals $\Sigma e_{ijk} a_1^i a_2^j a_3^k$, that is, with our summation convention,

$$|\alpha_s^r| = e_{ijk} a_1^i a_2^j a_3^k. \quad (18)$$

If we expand the determinant by rows, we easily see that we have the similar formula

$$|\alpha_s^r| = e^{ijk} a_i^1 a_j^2 a_k^3. \quad (19)$$

Let us examine

$$e_{ijk} a_i^1 a_j^2 a_k^3. \quad (20)$$

We shall first prove that this is completely skew-symmetric in r, s, t . The indices i, j, k are dummy indices and we have seen that the letter denoting a dummy index is quite

immaterial. Consequently we do not alter the sum if we substitute k for the dummy index i , and i for the dummy index k . Thus

$$e_{ijk} a_r^i a_s^j a_t^k = e_{kji} a_r^k a_s^j a_t^i = e_{kji} a_t^i a_s^j a_r^k = -e_{ijk} a_t^i a_s^j a_r^k,$$

since an interchange of two indices in e_{ijk} changes the sign. Hence (20) changes sign if r and t are interchanged, and the same result holds for any other two of the indices r, s, t , which proves our statement. Moreover (20) takes the value $|a_s^r|$ when r, s, t have the particular values 1, 2, 3 respectively, and we conclude from the equations (13) that

$$e_{ijk} a_r^i a_s^j a_t^k = |a_n^m| e_{rst}. \quad (21)$$

Similarly,

$$e^{ijk} a_i^r a_j^s a_k^t = |a_n^m| e^{rst}. \quad (22)$$

We have said that the upper indices denote the rows and the lower the columns. The equations (21) and (22) therefore show that an interchange of any two rows or columns changes the sign of the determinant but not its absolute value, and, in particular, if two rows or two columns are identical the determinant vanishes.

As an example of the power of this notation we may prove the well-known theorem on the multiplication of two determinants. Let the elements of the determinants be a_s^r and b_s^r . Then

$$\begin{aligned} a_s^r \vee b_s^r &= |a_s^r| e_{ijk} b_1^i b_2^j b_3^k \\ &= e_{mn\dot{p}} a_1^m a_2^n a_3^{\dot{p}} b_1^i b_2^j b_3^k \\ &= e_{mn\dot{p}} (a_1^m b_1^i) (a_2^n b_2^j) (a_3^{\dot{p}} b_3^k). \end{aligned}$$

Hence if we put

$$c_s^r = a_s^r b_s^m = a_1^r b_s^1 + a_2^r b_s^2 + a_3^r b_s^3,$$

we have

$$|a_s^r| \times |b_s^r| = e_{mn\dot{p}} c_1^m c_2^n c_3^{\dot{p}} = |c_s^r|,$$

and the formula for the product of two determinants is

$$|a_s^r| \times |b_s^r| = |a_m^r b_s^m|. \quad (23)$$

EXAMPLES.

1. Prove that $|\delta_s^r| = 1$.
2. If $a_m^r b_s^m = \delta_s^r$, then $|a_s^r|$ is the reciprocal of $|b_s^r|$.
3. If b_s^r is a double system such that $b_m^r b_s^m = \delta_s^r$, show that $|b_s^r| = \pm 1$.
4. Show that

$$\delta_{ijk}^rst a_m^i a_n^j a_p^k = |a_s^r| \delta_{mnp}^{rst}, \quad \text{and} \quad \delta_{ijk}^rst a_r^i a_s^j a_t^k = 3! |a_s^r|.$$

5. Work out the results corresponding to the above for determinants of the fourth order, by examining the case where the indices have the range of values 1, 2, 3, 4.

[We find that the ϵ -systems are now systems of the fourth order, and there are four types of Kronecker deltas.]

7. The cofactors of the elements in a determinant.

When the determinant (17), p. 10, is expanded in full it is obvious that any element such as a_i^r appears once in each of a certain number of terms of the expansion. The coefficient of a_i^r in this expansion we call *the cofactor of the element a_i^r in the determinant* and we denote it by A_r^i . Our object is to find an explicit expression for this cofactor.

Let us consider the element a_1^r . From (18), p. 10, the cofactor of a_1^r is seen to be $e_{rst} a_2^s a_3^t$, which can be written

$${}^{123} e_{rst} a_2^s a_3^t = \frac{1}{2!} e^{1jk} e_{rst} a_j^s a_k^t = \frac{1}{2!} \delta_{rst}^{1jk} a_j^s a_k^t,$$

that is,

$$A_r^1 = \frac{1}{2!} \delta_{rst}^{1jk} a_j^s a_k^t,$$

and a similar result holds for the cofactors A_r^2 and A_r^3 . Consequently we see that the cofactor of a_i^r is given by

$$A_r^i = \frac{1}{2!} \delta_{rst}^{ijk} a_j^s a_k^t. \quad (24)$$

Having obtained expressions for the cofactors A_r^i we are now in a position to obtain the expansion of the determinant in terms of the elements of any one row or of any one column. Let us consider $a_r^m A_m^i$. We have

$$\begin{aligned} a_r^m A_m^i &= \frac{1}{2!} \delta_{rst}^{ijk} a_r^m a_j^s a_k^t = \frac{1}{2!} e^{ijk} e_{mst} a_r^m a_j^s a_k^t \\ &= \frac{1}{2!} e^{ijk} e_{rst} a_r^m a_j^s a_k^t. \end{aligned}$$

Hence, if we denote the determinant by A ,

$$a_r^m A_m^i = A \delta_r^i. \quad \dots \dots \dots (25)$$

This formula gives the expansion of the determinant in terms of the elements of the r^{th} column. The student will have no difficulty in proving the similar relation

$$a_m^i A_r^m = A \delta_r^i, \quad (26)$$

which is the expansion in terms of the elements of the i^{th} row.

These formulae can be written in a slightly different form, which will be found useful later. If A is not zero, let us put

$$\alpha_r^i = A_r^i / A,$$

that is, α_r^i is the cofactor of a_r^i in A , divided by A . The formulae (25) and (26) now read

$$a_r^m \alpha_m^i = a_m^i \alpha_r^m = \delta_r^i. \quad (27)$$

EXAMPLES.

1. To express the cofactors of the reciprocal determinant in terms of the elements of the original determinant.

The reciprocal determinant of A is that whose elements are A_r^i . Let \bar{A} and \bar{A}_s^r denote the reciprocal determinant of A and the cofactors of its elements. We wish to express these quantities in terms of A and the elements of A . We have, by the rule for the multiplication of determinants,

$$\bar{A} A = |A_m^i a_r^m| = |A \delta_r^i| = A^3 | \delta_r^i | = A^3.$$

Hence, if $A \neq 0$,

$$\bar{A} = A^2.$$

Also, applying (25) to the reciprocal determinant, we obtain

$$\bar{A}_i^r A_s^i = \delta_s^r \bar{A}.$$

If we multiply this equation by a_j^s and sum s from 1 to 3, we get

$$\bar{A}_i^r \delta_j^i A = a_j^r \bar{A},$$

that is,

$$\bar{A}_j^r A = a_j^r \bar{A} = a_j^r A^2.$$

Therefore, if $A \neq 0$,

$$A_j^r = A a_j^r.$$

2. Prove that

$$e^{rst} A_r^i = e^{ijk} a_j^s a_k^t, \quad \text{and} \quad e_{ijk} A_r^i = e_{rst} a_j^s a_k^t.$$

3. Prove that

$$\delta_{mn}^i a_r^m a_s^n = a_r^i a_s^i - a_r^j a_s^j = \delta_{rst}^i A_k^i.$$

4. Show that

$$A = \frac{1}{3!} \delta_{rst}^i a_s^j a_j^t a_k^i, \quad a_r^m A_m^r = 3A.$$

5. Prove that

$$\frac{\partial A}{\partial a_r^i} = A_r^i.$$

8. Linear equations.

A system of three linear equations can be written

$$\begin{aligned} a_1^1 x^1 + a_2^1 x^2 + a_3^1 x^3 &= b^1, \\ a_1^2 x^1 + a_2^2 x^2 + a_3^2 x^3 &= b^2, \\ a_1^3 x^1 + a_2^3 x^2 + a_3^3 x^3 &= b^3. \end{aligned}$$

If we use our conventions regarding the indices these can be compressed into

$$a_m^r x^m = b^r. \quad (28)$$

We wish to find the solution of these equations.

Let us suppose that A is not zero. Then if we multiply (28) by A_r^i and sum r from 1 to 3, we shall have

$$A_r^i a_m^r x^m = A_r^i b^r,$$

or

$$A \delta_m^i x^m = A_m^i b^m. \quad (29)$$

But

$$\delta_m^i x^m = x^i,$$

and therefore, dividing (29) by A ,

$$\frac{A_m^i b^m}{A}. \quad (30)$$

It is easy to verify by direct substitution that (30) does satisfy the linear equations and we conclude that there is a unique solution.

If $A = 0$, but all the cofactors of its elements are not zero, we see from (29) that the equations are inconsistent unless $A_m^i b^m = 0$. When this relation is satisfied we have identically

$$A_r^i (a_s^r x^s - b^r) = 0,$$

which shows that the three equations (28) are linearly connected. Thus we have only to satisfy two of the equations and the third is also satisfied.

A particularly interesting case of (28) is where $b^r = 0$. We then see that the equations have the unique solution $x^r = 0$ unless A is zero. Also, if A is zero, the equations are not independent and we can find non-zero values of x^r to satisfy them. That is, $A = 0$ is the necessary and sufficient condition that the equations

$$a_m^r x^m = 0$$

should have a non-zero solution.

EXAMPLES.

1. If all the cofactors A_r^i are zero, show that $a_s^i a_r^j = a_r^i a_s^j$ and that the equations (28) are inconsistent unless $b^i a_r^i = b^j a_r^j$.

2. If the conditions of Ex. 1 are fulfilled, show that the three equations are only equivalent to one equation.

9. Corresponding formulae for the system a_{mn} .

We have developed the theory of determinants where the elements are of the form a_s^r . Obviously the results are exactly the same if the elements are written a_{rs} , both indices being subscripts. But if we are to arrange matters so that a dummy index will always appear once as a subscript and once as a superscript in each term, we shall require to modify our formulae a little. We shall only state the results and leave the student to prove the various formulae.

Writing out the determinant in full, we have

$$A \equiv |a_{mn}| \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \dots \dots \dots (31)$$

so that the first index denotes the row and the second index the column. We have

$$A = e^{ijk} a_{i1} a_{j2} a_{k3} = e^{ijk} a_{1i} a_{2j} a_{3k}, \dots \dots \dots (32)$$

and

$$e^{ijk} a_{ir} a_{js} a_{kt} = e^{ijk} a_{ri} a_{sj} a_{tk} = A e_{rst}. \dots \dots (33)$$

The cofactors of the elements are given by the formulae

$$A^{ir} = \frac{1}{2!} e^{ijk} e^{rst} a_{js} a_{kt} \dots \dots \dots (34)$$

where A^{ir} is the cofactor of a_{ir} in the determinant. Also the expansion of the determinant in terms of the elements of the r th column is

$$a_{mr} A^{mi} = A \delta_r^i, \dots \dots \dots (35)$$

whilst the expansion in terms of the elements of the r th row is

$$a_{rm} A^{im} = A \delta_r^i. \dots \dots \dots (36)$$

If A is not zero, let us put

$$\alpha^{ir} = A^{ir}/A,$$

so that α^{ir} is the cofactor of a_{ir} in A , divided by A . Then

$$a_{mr} \alpha^{mi} = a_{rm} \alpha^{im} = \delta_r^i. \dots \dots \dots (37)$$

If the system a_{mn} is symmetric we say that the determinant is *symmetric* and if a_{mn} is skew-symmetric the determinant is *skew-symmetric*.

EXAMPLES.

1. If a_{mn} is symmetric, show that A^{mn} is also symmetric.
2. If a_{mn} is skew-symmetric, show that in the third-order determinant A^{mn} is symmetric and A is zero.

3. Prove

$$e_{r m n} A^{i r} = e^{i j k} a_{j m} a_{k n}, \quad e_{r m n} A^{r i} = e^{i j k} a_{m j} a_{n k}.$$

4. Solve the equations $a_{r m} x^m = b_r$, following the analogy of § 8, p. 14.

5. Write down the formulae connected with the determinant formed by the double system a^{mn} .

10. Positive definite quadratic forms. The determinantal equation.

The quadratic form $a_{mn} x^m x^n$ is said to be *positive definite* if it vanishes only for the values $x^r = 0$ and is positive for all other real values of x^r .

If we are given two quadratic forms $a_{mn} x^m x^n$ and $b_{mn} x^m x^n$, where a_{mn} and b_{mn} are two *symmetric* double systems, we have often to consider the symmetric determinant

$$\begin{aligned} &\theta a_{11} - b_{11}, \quad \theta a_{12} - b_{12}, \quad \theta a_{13} - b_{13} \\ &\theta a_{21} - b_{21}, \quad \theta a_{22} - b_{22}, \quad \theta a_{23} - b_{23} \\ &\theta a_{31} - b_{31}, \quad \theta a_{32} - b_{32}, \quad \theta a_{33} - b_{33} \end{aligned}$$

This determinant we shall briefly denote by $|\theta a_{mn} - b_{mn}|$. We wish to prove that if $a_{mn} x^m x^n$ is a *positive definite quadratic form* then the roots of the determinantal equation

$$|\theta a_{mn} - b_{mn}| = 0 \quad . \quad . \quad . \quad . \quad . \quad (38)$$

are all real.

Let us suppose that $\alpha + i\beta$ is a root of (38). Then (p. 14) there exist non-zero quantities $\lambda^r + i\mu^r$ such that

$$[(\alpha + i\beta) a_{mn} - b_{mn}] (\lambda^n + i\mu^n) = 0.$$

Equating real and imaginary parts we have

$$\alpha a_{mn} \lambda^n - \beta a_{mn} \mu^n - b_{mn} \lambda^n = 0, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

$$\alpha a_{mn} \mu^n + \beta a_{mn} \lambda^n - b_{mn} \mu^n = 0. \quad . \quad . \quad . \quad . \quad . \quad (40)$$

If we multiply (40) by λ^m and (39) by μ^m , and sum m from 1 to 3, we obtain by subtraction

$$\beta (a_{mn} \lambda^m \lambda^n + a_{mn} \mu^m \mu^n) = (b_{mn} - b_{nm}) \lambda^m \mu^n = 0.$$

Now λ^r, μ^r are not all zero and $a_{mn} x^m x^n$ is positive definite. Hence $a_{mn} \lambda^m \lambda^n$ and $a_{mn} \mu^m \mu^n$ cannot both be zero, and we conclude that

$$\beta = 0,$$

which proves that the root is real. We have therefore proved our theorem.

If we multiply (39) by λ^m , (40) by μ^m , and sum m from 1 to 3, we get by addition

$$\alpha (a_{mn} \lambda^m \lambda^n + a_{mn} \mu^m \mu^n) = b_{mn} \lambda^m \lambda^n + b_{mn} \mu^m \mu^n.$$

If we are also given that $b_{mn}x^m x^n$ is a positive definite form then the right-hand side of this equation is positive and we conclude that α is positive. We have hence the following result:

If $a_{mn}x^m x^n$ and $b_{mn}x^m x^n$ are both positive definite forms then the roots of the determinantal equation (38) are all real and positive.

EXAMPLE.

If $a_{mn}x^m x^n$ is positive definite and b_{mn} is skew-symmetric, prove that the roots of (38) are either zero or purely imaginary. Deduce that the skew-symmetric determinant of odd order is zero.

EXAMPLES I.

1. Prove that if $a_{mn}x^m y^n = 0$ for arbitrary values of x^r and y^r , then $a_{mn} = 0$.

2. Prove the numerical relations:

$$\delta_{mnp}^{ijk} \delta_{rst}^{hlp} = \delta_{rst}^{ijk} \delta_{mnp}^{hlp}, \quad \delta_{mnp}^{ijk} \delta_{rst}^{hnp} = 2! \delta_{rst}^{ijk} \delta_m^h,$$

$$\delta_{mnp}^{ijk} \delta_{rst}^{mnp} = 3! \delta_{rst}^{ijk}.$$

3. Prove

$$\delta_{mn}^{rs} \quad \begin{matrix} \delta_{mn}^r & \delta_{mn}^s \\ \delta_{mn}^s & \delta_{mn}^r \end{matrix} \quad \delta_{mn}^{rst} \quad \begin{matrix} \delta_{mn}^r & \delta_{mn}^s & \delta_{mn}^t \\ \delta_{mn}^s & \delta_{mn}^r & \delta_{mn}^t \\ \delta_{mn}^t & \delta_{mn}^s & \delta_{mn}^r \end{matrix}$$

4. If a_{rs} and b_{rs} are both symmetric systems, such that

$$a_{rs}b_{tu} - a_{ru}b_{st} + a_{st}b_{ru} - a_{tu}b_{rs} = 0,$$

then $a_{rs} = kb_{rs}$.

5. If $a_{mn}x^m x^n = b_{mn}x^m x^n$ for arbitrary values of x^r , show that

$$a_{mn} + a_{nm} = b_{mn} + b_{nm};$$

hence, if a_{mn} and b_{mn} are symmetric, $a_{mn} = b_{mn}$.

6. If a_{mn} is either symmetric or skew-symmetric and $A = 0$, show that x^r can be chosen such that $A^{mn} = kx^m x^n$.

Hence deduce that if A^{11} , A^{22} , A^{33} are all zero, then every A^{mn} is zero. [The quantities x^r are those which satisfy $a_{mn}x^n = 0$.]

7. If $A = 0$ show that the solutions of the equations $a_{rm}x^m = 0$ satisfy

$$(x^1)^2 : (x^2)^2 : (x^3)^2 = A^{11} : A^{22} : A^{33}.$$

8. If α^{rs} are quantities such that $a_{rm}\alpha^{sm} = \delta_r^s$, prove that α^{rs} is the cofactor of a_{rs} in the determinant A , divided by A .

[We have $A^{rp}a_{rm}\alpha^{sm} = \delta_r^s A^{sp}$, that is, $A\alpha^{sp} = A^{sp}$.]

9. *Derivative of a determinant.* If α_r^s are all functions of the variables x^r , show that

$$\begin{aligned}\frac{\partial A}{\partial x^r} &= A_{m_1}^n \frac{\partial a_1^n}{\partial x^r} \\ \left[\frac{\partial A}{\partial x^r} &= e_{m_1 n} \frac{\partial a_1^{m_1}}{\partial x^r} a_2^n + e_{m_2 n} a_1^{m_1} \frac{\partial a_2^n}{\partial x^r} a_3^p + e_{m_3 n} a_1^{m_1} a_2^n \frac{\partial a_3^p}{\partial x^r} \right. \\ &\quad \left. = \frac{\partial a_1^{m_1}}{\partial x^r} A_{m_1}^1 + \dots \right].\end{aligned}$$

10. Let $h_{mn} = \theta a_{mn} - b_{mn}$, where a_{mn} and b_{mn} are two symmetric systems such that $a_{mn} x^m x^n$ is positive definite, and let H and H^{mn} denote the determinant $|h_{mn}|$ and the cofactors of its elements.

(i) Prove that $a_{mn} H^{mr} H^{ns} = H H'^{rs} - H' H'^{rs}$, where dashes denote differentiations with respect to θ .

(ii) Deduce that if θ is a double root of $H = 0$, then it is a root of each of the equations $H^{rs} = 0$.

11. Prove the following formulae connecting A , α_{mn} , α^{mn} :

$$\begin{aligned}\text{(i)} \quad & \frac{\partial \alpha_{mn}}{\partial a_{rs}} = \delta_m^r \delta_n^s, & \frac{\partial \alpha^{mn}}{\partial \alpha^{rs}} &= \delta_r^m \delta_s^n; \\ \text{(ii)} \quad & \frac{\partial \alpha^{mn}}{\partial a_{rs}} = -\alpha^{ms} \alpha^{rn}, & \frac{\partial \alpha_{mn}}{\partial \alpha^{rs}} &= -\alpha_{ms} \alpha_{rn}; \\ \text{(iii)} \quad & \frac{\partial (\log A)}{\partial a_{rs}} = \alpha^{rs}, & -\frac{\partial (\log A)}{\partial \alpha^{rs}} &= \alpha_{rs}; \\ \text{(iv)} \quad & \frac{\partial^2 A}{\partial a_{mn} \partial a_{rs}} = A (\alpha^{mn} \alpha^{rs} - \alpha^{ms} \alpha^{rn}), \\ & \frac{\partial^2 A}{\partial \alpha^{mn} \partial \alpha^{rs}} = A (a_{ms} a_{rn} - a_{mn} a_{rs}).\end{aligned}$$

12. If a_s^r satisfies the relations, $a_m^r a_s^m = \delta_s^r$, show that either

$$A = 1 \quad \text{and} \quad |a_s^r - \delta_s^r| = 0, \quad \text{or} \quad A = -1 \quad \text{and} \quad |a_s^r + \delta_s^r| = 0.$$

$$[|a_s^r \pm \delta_s^r| A = |a_m^r a_s^m \pm \delta_m^r a_s^m| = |\delta_s^r \pm a_s^r| = \pm |a_s^r \pm \delta_s^r|].$$

13. *Functional Determinants.* If y^1, y^2, y^3 are a set of functions of x^1, x^2, x^3 , we denote by $\frac{\partial(y^1, y^2, y^3)}{\partial(x^1, x^2, x^3)}$ the *functional determinant*, or *Jacobian*, of the y 's with respect to the x 's, that is,

$$\frac{\partial(y^1, y^2, y^3)}{\partial(x^1, x^2, x^3)} = \frac{\partial y^r}{\partial x^s}.$$

Prove that, if z^1, z^2, z^3 are functions of the y 's, then

$$\frac{\partial z^r}{\partial x^s} = \frac{\partial z^r}{\partial y^m} \frac{\partial y^m}{\partial x^n} = \frac{\partial z^r}{\partial y^s} \frac{\partial y^m}{\partial x^n}.$$

14. Prove that

$$\frac{\partial}{\partial x^r} \left\{ \log \frac{\partial y}{\partial x} \right\} = \frac{\partial^2 y^m}{\partial x^r \partial x^n} \frac{\partial x^n}{\partial y^m}.$$

15. Prove that

$$\frac{\partial(y^i, y^j, y^k)}{\partial(x^r, x^s, x^t)} = \left| \frac{\partial y^m}{\partial x^n} \right| \delta_{rst}^{ijk}.$$

16. *Bordered Determinants.* The determinant A is said to be bordered when an additional row and column are added. For example, such a determinant is

$$\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & u_1 & & \\ a_{21} & a_{22} & a_{23} & u_2 & a_{m\ n} & u_m \\ & & & & v_n & w \end{array}$$

Prove that

$$\begin{vmatrix} a_{m\ n} & u_m \\ v_n & w \end{vmatrix} = Aw - A^{m\ n} u_m v_n.$$

17. If $A^{m\ n}$ is the cofactor of $a_{m\ n}$ in the determinant $|a_{m\ n}|$, prove that

$$e_{p\ m\ n} e_{q\ r\ s} A^{p\ q} = 2(a_{m\ r} a_{n\ s} - a_{m\ s} a_{n\ r}).$$

18. If $a_{m\ n}$, $b_{m\ n}$ are two double systems, show that $|A^{m\ n} b_{m\ r} b_{n\ s}| = A^2 B^2$ and show that the cofactors of the elements in $|A^{m\ n} b_{m\ r} b_{n\ s}|$ are $A a_{m\ n} B^{m\ r} B^{n\ s}$.

CHAPTER II

TENSOR ANALYSIS

1. Linear transformations.

Let us suppose that the variables, x^1, x^2, x^3 , are transformed into a new set of variables, $\bar{x}^1, \bar{x}^2, \bar{x}^3$, by the linear transformation

$$\begin{aligned} \bar{x}^1 &= c_1^1 x^1 + c_2^1 x^2 + c_3^1 x^3, \\ \bar{x}^2 &= c_1^2 x^1 + c_2^2 x^2 + c_3^2 x^3, \\ \bar{x}^3 &= c_1^3 x^1 + c_2^3 x^2 + c_3^3 x^3, \end{aligned}$$

where the c 's are constants. Making use of our conventions about the indices, these equations are compressed into the single formula

$$\bar{x}^r = c_s^r x^s. \quad \dots \dots \dots (1)$$

We assume that $c = |c_s^r|$, the determinant of the transformation, is not zero. Let γ_s^r be the cofactor of c_s^r in the determinant c , divided by c . Then

$$c_m^r \gamma_s^m = \gamma_m^r c_s^m = \delta_s^r, \quad \dots \dots \dots (2)$$

and we can solve equations (1) for the x 's, giving

$$x^r = \gamma_m^r \bar{x}^m, \quad \dots \dots \dots (3)$$

which shows that the given transformation is *reversible*.

Again, if $c_s^r = \delta_s^r$, we have

$$\bar{x}^r = x^r,$$

which is the *identical* transformation.

If we transform first from the variables x^r to \bar{x}^r , according to equations (1), and then from \bar{x}^r to $\bar{\bar{x}}^r$, according to the equations

$$\bar{\bar{x}}^r = c_s'^r \bar{x}^s,$$

we see that the transformation from the original variables x^r to $\bar{\bar{x}}^r$ is given by

$$\bar{\bar{x}}^r = c_s''^r x^s,$$

where

$$c_s''^r = c_m'^r c_a^m.$$

This transformation is consequently also of the linear form.

A series of transformations is said to form a *group* if it satisfies the following conditions: (i) if the transformations from x^r to \bar{x}^r and from \bar{x}^r to $\bar{\bar{x}}^r$ are of the series, then the transformation from x^r to $\bar{\bar{x}}^r$ is also of the series; (ii) the series contains the identical and the reversible transformations.

Hence we see that the system of linear transformations forms a group.

EXAMPLES.

1. Show that

$$\gamma = |\gamma_s^r| = 1/c.$$

2. Prove that

$$|c_s''^r| = |c_s'^r| \cdot |c_s^r|$$

2. Invariants, contravariant and covariant vectors.

We now ask the question: — Given systems of different orders as defined in the previous chapter, how can these systems transform when the variables are transformed by a linear transformation?

This question leads us to the definition of a tensor. *A tensor is a system of numbers or functions of the type considered in the last chapter, whose components obey a certain law of transformation when the variables undergo a linear transformation (1).* Consequently there will be tensors of order zero, one, two, three, and so on, just as there are systems of these orders.

Let us first consider a system of order zero, that is, a single number or function. If this quantity has the *same* value in the

new variables \bar{x}^r as it has in the old variables x^r , it is called a *scalar*, or an *invariant*, or a *tensor of order zero*. Hence the quantity a is an invariant if

$$\bar{a} = a, \quad (4)$$

where \bar{a} is the value of the quantity in the new variables.

Next let us consider systems of order *one*. Now a simple case of a system of the first order is the system of variables itself, namely x^r . The components of this system become \bar{x}^r after the transformation, where

$$\bar{x}^r = c_s^r x^s.$$

Hence this gives one way in which a system of the first order may transform. Any such system, whose components transform in this way, is called a *contravariant tensor of order one* or, alternatively, a *contravariant vector*. Thus a^r is a contravariant vector if its transformed components, \bar{a}^r , are given by the equations

$$\bar{a}^r = c_s^r a^s, \quad (5)$$

when the variables undergo the linear transformation (1).

There is another way in which the components of a 1-system may transform. We have seen that the coefficients of a linear form in the x 's also form a system of order one. Thus the coefficients of the linear form $a_m x^m$ are the components of a system a_r . Let us now suppose that the a 's transform in such a way that the linear form $a_m x^m$ remains *invariant* under the transformation of variables. If we denote by \bar{a}_r the new components of the system a_r after the transformation, we have

$$\bar{a}_m \bar{x}^m = a_m x^m,$$

since the linear form is an invariant. It follows, from (3), that

$$\bar{a}_m \bar{x}^m = a_m \gamma_p^m \bar{x}^p.$$

Since a dummy index may be denoted by any letter, these equations can be written in the form

$$(\bar{a}_p - a_m \gamma_p^m) \bar{x}^p = 0.$$

If these relations are to be true for all values of the variables \bar{x}^r , we must have

$$\bar{a}_r = \gamma_r^m a_m. \quad (6)$$

This is obviously a different transformation law from (5), and a system of the first order, whose components transform in this way, is called a *covariant tensor of order one* or a *covariant vector*.

We have thus two types of tensors of the first order, and we distinguish these types by the position of the index. If the tensor is contravariant we use a superscript as index, and if it is covariant we use a subscript as index. In other words, *an upper index denotes contravariance, and a lower index covariance*.

EXAMPLES.

1. Show that

$$\bar{a}^r = \frac{\partial \bar{x}^r}{\partial x^s} a^s, \quad \bar{a}_r = \frac{\partial x^s}{\partial \bar{x}^r} a_s.$$

$$\left[c_s^r = \frac{\partial \bar{x}^r}{\partial x^s}, \quad \gamma_s^r = \frac{\partial x^r}{\partial \bar{x}^s} \right].$$

2. Prove the formulæ:

$$a^r = \gamma_m^r \bar{a}^m, \quad a_r = c_r^m \bar{a}_m.$$

3. If φ is an invariant function of the variables x^r , show that the system $\frac{\partial \varphi}{\partial x^r}$ is a covariant vector.

4. Show that the differentials, $d x^r$, of the variables form a contravariant vector.

3. Tensors of any order.

Our next step is to examine systems of orders higher than one, and to inquire how these may be conveniently transformed.

Let us consider a double system. A simple case of a double system is given by the product of two vectors or systems of the first order, and such a product may be of three different types: (i) the product of two contravariant vectors, (ii) the product of two covariant vectors, (iii) the product of a contravariant and a covariant vector.

(i) Thus $(a^r b^s)$ is a double system, which is the product of two contravariant vectors a^r, b^r . The components of this double system in the new variables are given by

$$(\bar{a}^r \bar{b}^s) = (c_m^r a^m) (c_n^s b^n) = c_m^r c_n^s (a^m b^n).$$

Hence a double system, a^{rs} , may transform according to the equations

$$\bar{a}^{rs} = c_m^r c_n^s a^{mn}. \quad \dots \dots \dots (7)$$

A system, whose components transform in this way, is called a *contravariant double tensor*, and it is distinguished by both indices being superscripts.

(ii) Again, $(a_r b_s)$ is a double system, which is the product of two covariant vectors a_r, b_r . Its new components after the transformation are

$$(\bar{a}_r \bar{b}_s) = (\gamma_r^m a_m) (\gamma_s^n b_n) = \gamma_r^m \gamma_s^n (a_m b_n).$$

Consequently a double system, a_{rs} , may transform according to the equations

$$\bar{a}_{rs} = \gamma_r^m \gamma_s^n a_{mn}, \quad (8)$$

and a system, transforming in this way, is called a *covariant double tensor*. It is distinguished by both indices being subscripts.

(iii) Lastly, $(a^r b_s)$ is a double system formed by the multiplication of a contravariant vector, a^r , and a covariant vector, b_r . The transformation of this system is in accordance with the equations

$$(a^r \bar{b}_s) = (c_m^r a^m) (\gamma_s^n b_n) = c_m^r \gamma_s^n (a^m b_n).$$

Hence a double system, a_s^r , can transform according to the law

$$\bar{a}_s^r = c_m^r \gamma_s^n a_n^m. \quad (9)$$

Such a double system is called a *mixed double tensor* and is distinguished by one upper and one lower index.

It will be an easy matter for the student to generalise our definitions to include tensors of any order. As an example, let us consider a triple tensor or tensor of the third order. Obviously one such system is obtained by multiplying together three vectors. As a type, let us take the triple system $(a^r b_s c_t)$, obtained by multiplying together a contravariant vector, a^r , and two covariant vectors, b_r, c_r . Its transformation law is

$$(\bar{a}^r \bar{b}_s \bar{c}_t) = (c_m^r a^m) (\gamma_s^n b_n) (\gamma_t^p c_p) = c_m^r \gamma_s^n \gamma_t^p (a^m b_n c_p).$$

Therefore a triple system, a_{st}^r , may transform according to the equations

$$\bar{a}_{st}^r = c_m^r \gamma_s^n \gamma_t^p a_{np}^m. \quad (10)$$

Any triple system, a_{st}^r , whose components transform in this way, is called a *mixed triple tensor*, contravariant with respect to the upper index and covariant with respect to both lower indices.

As a mnemonic to remember the laws of transformation of tensors we have the following:

(i) The typical contravariant vector is x^r , and all contravariant vectors transform in the same way as x^r .

- (ii) The typical covariant vector is $\frac{\partial \varphi}{\partial x^r}$, where φ is an invariant function of the variables x^r , and all covariant vectors transform like this.
- (iii) The typical tensor of any order is a product of vectors formed as follows:—for every contravariant (or upper) index we have a contravariant vector as a factor of the product, and for every covariant (or lower) index we have a covariant vector as a factor.

We notice that each component of a tensor in the new variables is a linear combination of the components in the old variables. Consequently, if *all* the components of a tensor are zero in the original system of variables, they are also all zero in the new variables. This is the most important property possessed by tensors.

Another fact to be noted is the *arbitrariness of tensors*. We may take as the components of a tensor in one set of variables any set of quantities of the requisite number and define their components in any other system of variables by the linear equations expressing the law of transformation for that particular tensor. Thus, if we wish to obtain a triple tensor contravariant in one index and covariant in the other two, we can take any set of 3^3 quantities a''_{st} as the components of the tensor in the variables x^r , and the components a'_{st} in the variables \bar{x}^r will then be given by the equations (10).

EXAMPLES.

1. Write down the transformation laws for the different types of tensors of the fourth order.

2. Show that $\frac{\partial^2 \varphi}{\partial x^r \partial x^s}$ is a double covariant tensor, φ being a function of the x 's.

3. If we have an equation of the form $a''_{st} = b''_s c_t$ between the tensors a''_{st} , b''_s , c_t , in one system of variables, show that the same relations between the components hold in every system of variables.

4. Show that δ''_{st} , δ^{rs}_{mn} , δ^{rst}_{mnp} are tensors, that is, the Kronecker deltas are tensors.

[These results follow from equations (2), p. 19.]

5. If a tensor has its components in one system of variables symmetric (or skew-symmetric), show that its components in every other system of variables are also symmetric (or skew-symmetric). Such a tensor is called *symmetric* (or *skew-symmetric*).

4. Addition, multiplication and contraction of tensors.

We have seen in the last chapter that the addition of the corresponding components of two systems of the same order

and type gives another system of that order and type. Thus, if we have the two triple systems, a_{st}^r and b_{st}^r , then

$$c_{st}^r = a_{st}^r + b_{st}^r \dots \dots \dots (11)$$

defines another system, called the *sum* of the original two. We shall now show that, if a_{st}^r and b_{st}^r are triple tensors, so also is c_{st}^r . For the components in the new system of variables are given by

$$\begin{aligned} \bar{c}_{st}^r &= \bar{a}_{st}^r + \bar{b}_{st}^r = c_m^r \gamma_s^n \gamma_t^p a_{n,p}^m + c_m^r \gamma_s^n \gamma_t^p b_{n,p}^m \\ &= c_m^r \gamma_s^n \gamma_t^p (a_{n,p}^m + b_{n,p}^m) = c_m^r \gamma_s^n \gamma_t^p c_{n,p}^m, \end{aligned}$$

which proves our statement. Thus the operation of *addition* applied to two tensors gives another tensor of the same order and type. Subtraction is of course included as a particular case of addition. It follows immediately from these results that the equations

$$a_s^r = b_s^r$$

form a *tensor equation*, that is, if they are true in one system of variables, they are true in all systems; for these equations merely state that the tensor ($a_s^r - b_s^r$) has all its components zero in one system of variables and therefore in all others.

We have also found that, when we multiply each component of one system by every component of another system, we obtain a system whose order is equal to the sum of the orders of the two given systems. Thus if a_{st}^r is a 3-system and b_q^p is a 2-system, then c_{qst}^{pr} , defined by the equations

$$c_{qst}^{pr} = b_q^p a_{st}^r \dots \dots \dots (12)$$

is a 5-system, which is called the *product* of the two original systems. We leave it as an exercise to the student to prove that, if a_{st}^r and b_q^p are tensors of the types indicated by their indices, then c_{qst}^{pr} is a 5-tensor with its indices in the correct positions as subscripts and superscripts. Hence the process of *multiplication* applied to tensors produces other tensors. It is to be noted that in the operations of addition and multiplication a subscript remains a subscript and a superscript remains a superscript.

Finally there is the operation of *contraction* to be considered, and we shall see that it also produces tensors when it is performed on tensors. Let us take a mixed tensor, say of the third order for

simplicity, and let us denote it by a_{st}^r . If we make the indices r and t the same, so that the system becomes a_{sr}^r , and if we remember that the dummy index r is to be summed from 1 to 3, we see that the new system has its order reduced by two — by one covariant and one contravariant index. We shall show that the contracted system is also a tensor. Denoting, as usual, the transformed components of the systems by barred letters, we have

$$\bar{a}_{st}^r = c_m^r \gamma_s^n \gamma_t^p a_{np}^m.$$

When we contract r and t , this gives us

$$\bar{a}_{sr}^r = c_m^r \gamma_s^n \gamma_r^p a_{np}^m = \delta_m^p \gamma_s^n a_{np}^m = \gamma_s^n a_{np}^p,$$

which shows that $\bar{b}_s = a_{sr}^r$ is a covariant vector.

We therefore see that the three elementary operations of addition, multiplication and contraction are all tensor operations, and any combination of these operations on given tensors will obviously produce new tensors. Hence we can often recognise the tensorial character of a system by observing that it is formed by a combination of these operations on known tensors. For example, if a_{mn} and b^{mn} are two tensors, then $a_{mn}b^{mn}$ is an invariant, for it is formed by multiplying the two tensors together and then contracting both pairs of indices.

EXAMPLES.

1. Show that $a_{st}^r b_r^p$ is a tensor of the third order.
2. Show that the equations, $(a_{st}^r + b_{st}^r) x^t = k_s^r$, form a tensorial set of equations.

5. The quotient law of tensors.

Let us suppose that we are given a set of equations of the form

$$A_{st}^r B^{st} = C^r,$$

connecting the systems A_{st}^r , B^{st} , C^r . If we know that A_{st}^r and B^{st} are tensors of the types indicated by their indices, we can immediately deduce that C^r is a tensor of the type indicated, since it is obtained by multiplying the other two systems together and contracting. Now it is of the greatest importance to be able to recognise tensors by the converse method; that is, if we know that C^r and B^{st} are tensors, can we deduce that A_{st}^r is a tensor?

In this connection we have the following result, which we are stating in a simple form but which can obviously be generalised to apply to systems of any order:

If we are given the relation

$$A(r, s, t) B^{st} = C^r, \dots \dots \dots (13)$$

where C^r is known to be a tensor and B^{st} is an arbitrary tensor, then $A(r, s, t)$ is a tensor and is to be represented by A_{st}^r .

To prove this theorem, let us give the variables x^r the linear transformation (1), p. 19, to the variables \bar{x}^r , and let barred letters refer to the new set of variables. We have

$$\bar{A}(r, s, t) \bar{B}^{st} = \bar{C}^r = c_m^r C^m = c_m^r A(m, n, p) B^{np},$$

and

$$B^{np} = \gamma_s^n \gamma_t^p \bar{B}^{st}.$$

Therefore

$$[\bar{A}(r, s, t) - c_m^r \gamma_s^n \gamma_t^p A(m, n, p)] \bar{B}^{st} = 0, \dots \dots \dots (14)$$

But B^{st} is a perfectly arbitrary tensor and hence so also is \bar{B}^{st} . Consequently all the coefficients of \bar{B}^{st} in (14) must be zero, that is,

$$\bar{A}(r, s, t) = c_m^r \gamma_s^n \gamma_t^p A(m, n, p),$$

which shows that $A(r, s, t)$ is a triple tensor and is correctly represented by A_{st}^r .

It is important to notice that the tensor B^{st} must be completely arbitrary, and must not be restricted to be either symmetric or skew-symmetric. What is essential is that we shall be able to give an arbitrary value to any one component and make all the others zero.

There is a particular case of our quotient theorem which is of some importance. Let us suppose that x^r, y^r, z_r are three arbitrary vectors, of which the first two are contravariant and the last covariant. If we know that $A_{st}^r x^s y^t z_r$ is an invariant, our quotient theorem states that A_{st}^r is a triple tensor, that is, *the coefficients of a multilinear form in the variables x^r, y^r, z_r form a tensor*. This result is sometimes taken as a *definition* of tensors, and it is easy to deduce from this definition the laws of transformation of tensors under a linear transformation of variables.

EXAMPLES.

1. If we are given the relation (13), where B^{st} is an arbitrary *symmetric* tensor, prove that $\{A(r, s, t) + A(r, t, s)\}$ is a tensor. Hence deduce that, if $A(r, s, t)$ is symmetrical in the indices s, t , then $A(r, s, t)$ is a tensor.

2. If we are given the relation (13), where B^{st} is an arbitrary *skew-symmetric* tensor, prove that $\{A(r, s, t) - A(r, t, s)\}$ is a tensor, and, if $A(r, s, t)$ is skew-symmetrical in s, t , then $A(r, s, t)$ is a tensor.

3. If $a_{mn} x^m x^n$ is an invariant, and a_{mn} is symmetric, show that a_{mn} is a tensor.

4. From the results of Ex. 3, p. 9, deduce that the Kronecker deltas are all tensors.

5. If a_{mn} is a tensor, show that a^{mn} , the cofactor of a_{mn} in $|a_{mn}|$ divided by $|a_{mn}|$, is also a tensor.

[Let k^r be an arbitrary vector, then $k_s = a_{sr} k^r$ is also an arbitrary vector. Now $\alpha^{st} k_s = k^t$, and the quotient law applies.]

6. Relative or weighted tensors.

We shall now extend the definition of a tensor, and we shall call the new tensors *relative* or *weighted tensors* to distinguish them from the type of tensor we have been considering up to the present.

Thus far a set of quantities, a_{st}^r , is said to be a tensor if it transforms according to the equations

$$\bar{a}_{st}^r = c_m^r \gamma_s^n \gamma_t^p a_{np}^m. \quad \dots \quad (15)$$

Here the equations of transformation from the variables x^r to the variables \bar{x}^r are

$$x^r = \gamma_s^r \bar{x}^s, \quad \dots \quad (16)$$

and the determinant of this transformation,

$$\gamma = |\gamma_j^i|, \quad \dots \quad (17)$$

is not zero.

We extend the term tensor to a system of the type, a_{st}^r , which transforms in accordance with the equations

$$\bar{a}_{st}^r = (\gamma)^M c_m^r \gamma_s^n \gamma_t^p a_{np}^m, \quad \dots \quad (18)$$

where $(\gamma)^M$ means the determinant γ raised to the power M . The equations (18) differ from (15) only in the factor $(\gamma)^M$ on the right-hand side. In fact the equations (15) are only a particular case of the equations (18), namely, where $M=0$, so that our new definition of tensor includes the old. The set of quantities, which satisfy equations (18), we call a *relative tensor* and we define its *weight* as M . The tensors we have been discussing previously are all of weight *zero*, and are referred to as *absolute tensors* when we wish to distinguish them. We shall usually

understand by the word tensor an absolute tensor unless the contrary is explicitly stated. Relative tensors of orders one and zero are called respectively *relative vectors* and *relative invariants*, just as for absolute tensors.

It is important to note that, if we know an invariant of weight one, we can immediately turn any relative tensor into an absolute tensor. For let us suppose that the quantity K is an invariant of weight one, and that α_{st}^r is a tensor of weight M . The transformed quantity \bar{K} is given by

$$\bar{K} = \gamma K,$$

and therefore

$$\bar{K}^M = (\gamma)^M K^M,$$

where we have raised each side to the M^{th} power. In other words K^M is an invariant of weight M . Also

$$\bar{\alpha}_{st}^r = (\gamma)^M \alpha_{st}^r \gamma_s^m \gamma_t^n \alpha_{mn}^p.$$

Consequently

$$(\bar{K}^{-M} \bar{\alpha}_{st}^r) = \alpha_{st}^r \gamma_s^m \gamma_t^n (K^{-M} \alpha_{mn}^p),$$

that is, $(K^{-M} \alpha_{st}^r)$ is an absolute tensor. In the sequel we shall transform relative tensors into absolute tensors in this way wherever it is possible, and consequently we shall be dealing almost always with absolute tensors.

We leave the student to verify the following properties of relative tensors, which are analogous to those possessed by absolute tensors:

(i) *Addition*. If we add together the corresponding components of two relative tensors of the same order and of the same weight M , we obtain a tensor of weight M , which is called the *sum* of the two original tensors.

(ii) *Multiplication*. If we multiply each component of a relative tensor of weight M and order m by every component of a relative tensor of weight N and order n , we obtain a tensor of weight $M + N$ and order $m + n$, which is called the *product* of the two original tensors.

(iii) *Contraction*. The operation of contracting an upper and a lower index results in a relative tensor of the same weight as that of the original tensor but with the order diminished by two.

(iv) *The quotient law*. If we are given the relation

$$A(r, s, t) B^{st} = C^r,$$

where C^r is known to be a relative tensor of weight M and B^{st} is an arbitrary relative tensor of weight N , then $A(r, s, t)$ is a relative tensor of weight $M - N$, representable by A_{st}^r . The proofs of these results are exactly the same as those already given for absolute tensors, the only modification required being the introduction of the additional factor in the laws of tensor transformation.

EXAMPLES.

1. Prove that e_{rst} and e^{rst} are relative tensors of weights -1 and 1 respectively.

From the formulae (21) and (22), p. 11, we have

$$e^{rst} = e^{mnp} c_m^r c_n^s c_p^t,$$

and

$$\gamma e_{rst} = e_{mnp} \gamma_r^m \gamma_s^n \gamma_t^p.$$

We know that $c = 1/\gamma$, and hence these equations can be written

$$e_{rst} = (\gamma)^{-1} \gamma_r^m \gamma_s^n \gamma_t^p e_{mnp},$$

$$e^{rst} = \gamma c_m^r c_n^s c_p^t e^{mnp}.$$

Now the e -systems have exactly the same components in every system of variables, namely, $0, 1, -1$ according to the arrangement of the indices. Therefore the e -systems in the set of variables \bar{x}^r will still be denoted by e_{rst} and e^{rst} . Consequently these last two equations give the laws by which the e -systems transform, and we see immediately that e_{rst} is a triple covariant tensor of weight -1 and e^{rst} is a triple contravariant tensor of weight 1 .

2. Deduce from Ex. 1. that the Kronecker deltas are *absolute tensors*.

3. If all the components of a relative tensor vanish in one system of variables, show that they vanish in every other system.

4. If all the components of two relative tensors of the same order and weight are equal in one system of variables, show that they are equal in every system.

[Two tensors whose components are equal in any system of variables are said to be *equal*.]

5. If a_{st}^r is a tensor depending on a variable parameter t , show that the derivative of the tensor with respect to t is a tensor of the same order and weight.

[This result is proved by differentiating the equations which give the laws of transformation of tensors, and is due to the fact that the c_n^m are constants.]

6. If a_{st}^r is a tensor, whose components are functions of the variables x^r , show that the partial derivatives $\frac{\partial a_{st}^r}{\partial x^u}$ form a tensor whose order is increased by one covariant index.

[This result is also due to the fact that the c_n^m are constants.]

7. If a_{st}^r is a relative tensor of weight M , show that

$$x_{st}^r = c^M \gamma_m^r c_n^s c_l^t \bar{a}_{nlp}^m.$$

7. General functional transformations.

Up to the present we have considered only *linear* transformations of variables, and we have defined tensors of different orders with respect to such transformations. We shall now

consider more general transformations of our variables, and inquire if we can extend our definitions of tensors to include these new transformations.

Let us suppose that the transformation from the variables x^r to \bar{x}^r is the *general functional* one, that is, the \bar{x}^r are arbitrary functions of the x^r . The formulae of transformation will be therefore of the form

$$\bar{x}^r = f^r(x^1, x^2, x^3), \quad (19)$$

where f^1, f^2, f^3 are three arbitrary functions of the x 's. We shall assume that the functions we deal with possess derivatives up to any order required, and also that the transformation (19) is reversible. Hence we have the inverse transformation

$$x^r = g^r(\bar{x}^1, \bar{x}^2, \bar{x}^3). \quad (20)$$

The necessary and sufficient condition that the transformation may be reversible is that the functional determinant,

$$c = \frac{|\partial \bar{x}^r}{\partial x^s} \quad (21)$$

should not be zero. This determinant is often referred to briefly as the *determinant of the transformation*. We note that the linear transformation (1), p. 19, is only a particular case of the functional transformation (19), namely, the case where the functions on the right-hand side of (19) are linear homogeneous functions of x^r .

If we take the differentials of (19) and (20), we obtain

$$d\bar{x}^r = \frac{\partial \bar{x}^r}{\partial x^s} dx^s, \quad (22)$$

$$dx^r = \frac{\partial x^r}{\partial \bar{x}^s} d\bar{x}^s. \quad (23)$$

If we write

$$c_s = \frac{\partial \bar{x}^r}{\partial x^s}, \quad \gamma_s^r = \frac{\partial x^r}{\partial \bar{x}^s}, \quad (24)$$

we see that equations (22) and (23) become

$$d\bar{x}^r = c_s^r dx^s, \quad dx^r = \gamma_s^r d\bar{x}^s.$$

In other words, *the transformation of the differentials is a linear one*, similar to (1) and (3), p. 19. The essential difference is that now the c 's and γ 's are not in general constants but are functions of x^r , and hence of \bar{x}^r also.

EXAMPLES.

1. Show that the system of transformations (19) forms a group.
2. Prove that

$$\frac{\partial \bar{x}^r}{\partial x^m} \frac{\partial x^m}{\partial \bar{x}^s} \delta_s^r = \frac{\partial x^r}{\partial \bar{x}^m} \frac{\partial \bar{x}^m}{\partial x^s}.$$

3. Show that

$$\frac{\partial \bar{x}^r}{\partial x^s} = 1, \quad \frac{\partial x^r}{\partial \bar{x}^s}.$$

4. Show that we have the general linear transformation of variables if we take the quantities $\frac{\partial \bar{x}^r}{\partial x^s}$ to be constants.

8. Tensors with respect to the general functional transformation.

We shall next define tensors with respect to the system of functional transformations. If we are given a system of functions of a certain order, we say that *it forms a tensor with respect to the transformation (19) when it is a tensor with respect to the linear transformation (22), (23) between the differentials of the old and new variables.*

Thus a system a_{st}^r is a tensor of weight M for the general transformation when its components \bar{a}_{st}^r in the new variables satisfy the relations*

$$\bar{a}_{st}^r = \left| \frac{\partial x^i}{\partial \bar{x}^j} \right|^M \frac{\partial \bar{x}^r}{\partial x^m} \frac{\partial x^m}{\partial \bar{x}^s} \frac{\partial x^p}{\partial \bar{x}^t} a_{np}^m, \quad \dots \quad (25)$$

for all values of the variables. As before, tensors of orders one and zero are called respectively vectors and invariants, and tensors of weight zero are referred to as absolute tensors.

Since linear transformations are only particular cases of functional transformations, we see that all systems which are tensors with respect to functional transformations are also tensors with respect to linear transformations. But the converse is not true, for there exist systems which are tensors with respect to linear transformations and are not tensors with respect to functional transformations. For example, the variables themselves are vectors under a linear change but are not vectors under the change (19).

* It may be noted as a useful mnemonic that the free indices r, s, t on the right of (25) are attached to the barred letters.

From the nature of the formulae (25), which give the definition of a tensor, we see that all processes or operations that are purely algebraic in character apply equally to tensors of the general transformation and tensors of the linear transformation. Consequently all the important results regarding addition, multiplication, contraction and the quotient law of tensors can be taken over without alteration to apply to the more general type of tensor.

We shall mostly confine our attention to absolute tensors, and it will be understood that the tensor is an absolute one unless the contrary is explicitly stated. The law of transformation for the absolute tensor a_{st}^r is

$$\bar{a}_{st}^r = \frac{\partial \bar{x}^r}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^s} \frac{\partial x^p}{\partial \bar{x}^t} a_{np}^m \dots \dots \dots (26)$$

The student will easily verify the following:

(i) The typical contravariant vector is dx^r .

(ii) The typical covariant vector is $\frac{\partial \varphi}{\partial x^r}$, where φ is an absolute invariant.

(iii) The typical tensor of any order is formed by multiplying together the requisite number of covariant and contravariant vectors, as described in § 3, p. 22.

We must remark that if a_{st}^r is a tensor function of the variables, then its derivatives with respect to x^r do not form a tensor. The result which we found (p. 30) to be true for tensors of the linear transformation holds no longer, since the quantities $\frac{\partial \bar{x}^r}{\partial x^s}$ are not now constants but are functions of the variables. Indeed, if we differentiate (26), we find

$$\begin{aligned} \frac{\partial \bar{a}_{st}^r}{\partial \bar{x}^u} &= \frac{\partial \bar{x}^r}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^s} \frac{\partial x^p}{\partial \bar{x}^t} \frac{\partial x^q}{\partial \bar{x}^u} \frac{\partial a_{np}^m}{\partial x^q} + \frac{\partial \bar{x}^r}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^s} \frac{\partial^2 x^p}{\partial \bar{x}^t \partial \bar{x}^u} a_{np}^m \\ &+ \frac{\partial \bar{x}^r}{\partial x^m} \frac{\partial^2 x^n}{\partial \bar{x}^s \partial \bar{x}^u} \frac{\partial x^p}{\partial \bar{x}^t} a_{np}^m + \frac{\partial^2 \bar{x}^r}{\partial x^m \partial \bar{x}^u} \frac{\partial x^n}{\partial \bar{x}^s} \frac{\partial x^p}{\partial \bar{x}^t} \frac{\partial x^q}{\partial \bar{x}^u} a_{np}^m, \end{aligned}$$

which shows that the derivatives do not satisfy the law of tensor transformation. This problem of forming tensors by differentiation we postpone to Part III (p. 146).

EXAMPLES.

1. Write down the transformations for tensors of rank one, two, three, giving the different types of each order.

2. If φ is an absolute invariant, show that $\frac{\partial \varphi}{\partial x^r}$ is a covariant vector.

3. If a_{mn} is symmetric and $a_{mn} dx^m dx^n$ is an invariant, show that a_{mn} is a covariant tensor.

EXAMPLES II.

1. If a_s^h is an absolute tensor, show that $|a_s^r|$ is an absolute invariant.

$$[|\bar{a}_s^r| = |c_m^r \gamma_s^m a_n^m| = |c_m^r| \cdot |\gamma_s^m a_n^m| = |c_m^r| \cdot |\gamma_s^m| \cdot |a_n^m| = |a_n^m| \cdot |.]$$

2. If a_{mn} is an absolute covariant tensor, show that $|a_{mn}|$ is an invariant of weight 2.

3. If a^{mn} is an absolute contravariant tensor, show that $|a^{mn}|$ is an invariant of weight -2.

4. If $\lambda_{(1)}^r, \lambda_{(2)}^r, \lambda_{(3)}^r$, are three absolute contravariant vectors, show that $|\lambda_{(i)}^r|$ is an invariant of weight -1.

[When we are dealing with several tensors we may distinguish one tensor from another by an index, which we enclose in brackets to show that it does not denote tensorial character.]

5. If $\lambda_r^{(1)}, \lambda_r^{(2)}, \lambda_r^{(3)}$, are three absolute covariant vectors, show that $|\lambda_r^{(i)}|$ is an invariant of weight 1.

6. Deduce from Exs. 4 and 5 that e_{rst} and e^{rst} are relative tensors of weights -1 and 1 respectively (cf. p. 30).

$$[|\lambda_{(i)}^r| = e_{rst} \lambda_{(1)}^r \lambda_{(2)}^s \lambda_{(3)}^t, \text{ and } |\lambda_r^{(i)}| = e^{rst} \lambda_r^{(1)} \lambda_r^{(2)} \lambda_r^{(3)}.]$$

7. If a_{mn} is an absolute tensor such that $|a_{mn}| = A$ is positive and not zero, prove that the two systems

$$\sqrt{A} e_{rst}, \quad e^{rst} / \sqrt{A},$$

are absolute tensors.

8. Prove that the cofactors of the determinant $|a_s^r|$ are absolute tensors, where a_s^r is an absolute tensor.

9. If α_s^r is the cofactor of a_s^r in $|a_s^r|$, divided by $|a_s^r|$, prove that α_s^r is an absolute tensor.

10. Show that the cofactors of the determinant $|a_{mn}|$ are all relative tensors of weight 2, where a_{mn} is an absolute covariant tensor.

11. If $A = |a_{mn}|$ and α^{mn} is the cofactor of a_{mn} in A , divided by A , prove that α^{mn} is an absolute contravariant tensor.

12. If $\lambda_r^{(i)}$ is the cofactor of $\lambda_{(i)}^r$ in the determinant of Ex. 4, divided by the determinant, prove that $\lambda_r^{(1)}, \lambda_r^{(2)}, \lambda_r^{(3)}$ are three absolute covariant vectors.

13. If $|a_{mn} - \theta g_{mn}|$ vanishes for a value θ_0 of θ in one set of variables, where a_{mn}, g_{mn} are absolute tensors, show that $|\bar{a}_{mn} - \theta \bar{b}_{mn}|$ in the new variables also vanishes for $\theta = \theta_0$. In other words the roots of $|a_{mn} - \theta g_{mn}|$ are invariants.

14. If all the cofactors of the elements of $|a_{mn}|$ vanish in one set of variables then they all vanish in every other set of variables.

15. If $\varphi = a_{mn} x^m x^n$, prove that $\frac{\partial^2 \varphi}{\partial x^m \partial x^n} = a_{mn} + a_{nm}$. Hence deduce that if φ is an invariant then $(a_{mn} + a_{nm})$ is a covariant tensor under linear transformations.

PART II

Algebraic Geometry

CHAPTER III

RECTILINEAR COORDINATES

1. Coordinates and tensors.

If we take three orthogonal planes intersecting at a point O , we know that the position of any point P in space is uniquely determined by the three perpendiculars from P on these planes, each with its proper sign. Let OY_2Y_3 , OY_3Y_1 and OY_1Y_2 be the three planes, and let the three perpendiculars be (y^1, y^2, y^3) . These perpendiculars are called the coordinates of the point P relative to a Cartesian orthogonal system of axes, the three planes are the coordinate planes, the three straight lines OY_1, OY_2, OY_3 are the coordinate axes, and O is the origin of coordinates.

Let us now make a general linear transformation to new variables (x^1, x^2, x^3) , according to the equations

$$\begin{aligned}x^1 &= \alpha_1^1 y^1 + \alpha_2^1 y^2 + \alpha_3^1 y^3, \\x^2 &= \alpha_1^2 y^1 + \alpha_2^2 y^2 + \alpha_3^2 y^3, \\x^3 &= \alpha_1^3 y^1 + \alpha_2^3 y^2 + \alpha_3^3 y^3.\end{aligned}$$

Remembering our conventions as to indices, these equations may be written more briefly

$$x^r = \alpha_r^s y^s. \quad \dots \quad (1)$$

We assume that the determinant $|\alpha_s^r|$ is not zero, and consequently we can solve (1) for the y 's in terms of the x 's, giving

$$y^r = \alpha_s^r x^s. \quad (2)$$

We conclude that to every point in space there corresponds a unique set of values of x^r and, conversely, to each set of values of x^r there corresponds a unique point of space. The variables x^r can therefore be looked upon as the coordinates of a point in space. All such coordinates, which are obtained by a linear transformation of the y 's, we shall call *rectilinear coordinates*.

It is a well-known result that the equation of a plane through the origin in orthogonal Cartesian coordinates is a linear equation of the form

$$\alpha_m y^m = 0. \quad (3)$$

Consequently we see that $x^1 = 0$, $x^2 = 0$ and $x^3 = 0$ are the equations of three planes through O . These planes we shall call the *coordinate planes of the system* x^r . Also if we take the intersections of these planes in pairs we get three straight lines through O , which we call the *coordinate axes of the system*. Obviously the equations of the three coordinate axes are the three pairs of equations

$$\begin{array}{lll} x^2 = 0 & x^3 = 0 & 0 \\ x^3 = 0 & x^1 = 0 & x^2 = 0. \end{array}$$

We shall denote these axes by OX_1 , OX_2 , OX_3 respectively. Lastly we call the *origin of the system* the point whose coordinates are $x^1 = 0$, $x^2 = 0$, $x^3 = 0$, and we see from (2) that the origin of the system x^r is the same as that of the Cartesian system. In other words, *the origin is unchanged by the transformation* (1).

If we have two systems of rectilinear coordinates x^r and \bar{x}^r , it is obvious from the relations (1) and (2) that these two sets of variables are connected by a linear relation

$$\bar{x}^r = c_s^r x^s, \quad (4)$$

where the c 's are constants; and that this can be solved for x^r in terms of \bar{x}^r , namely

$$x^r = \gamma_s^r \bar{x}^s. \quad (5)$$

Thus the change from one set of rectilinear coordinates to another is equivalent to a linear transformation of variables. We can therefore define *tensors, vectors and invariants with respect to a transformation of rectilinear coordinates*. For example, A_{st}^r is a triple tensor if its components in the new system of coordinates satisfy the equations

$$\bar{A}_{st}^r = c_n^r \gamma_s^n \gamma_t^p A_{np}^m.$$

EXAMPLE.

Show that the fundamental Cartesian system y^r is one of the set of rectilinear coordinate systems.

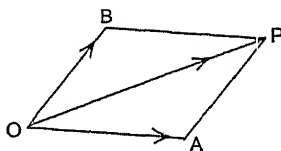
2. Contravariant vectors and displacements.

We have seen that with respect to a linear transformation of variables the variables x^r themselves form a contravariant vector. In other words, the coordinates of any point are the components of a contravariant vector. We shall now show that the converse is also true, that is, to every contravariant vector there corresponds a point whose coordinates in any coordinate system are the components of the vector in that system. Let a^r be a given contravariant vector and let A be the point whose coordinates are a^1, a^2, a^3 in any *one* system of coordinates. If x^r are the coordinates of A , then

$$x^r = a^r$$

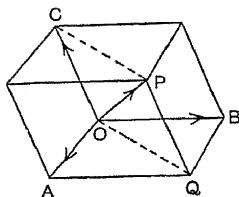
is a vector equation which is true in one coordinate system. It is therefore true in every coordinate system, and our result is proved. Instead of fixing attention on the point A , we may speak of the displacement OA , and we have the following geometrical interpretation of contravariant vectors: *to every contravariant vector there corresponds a displacement OA from the origin.*

Let us interpret geometrically the addition of two contravariant vectors a^r and b^r . Corresponding to these vectors we have two displacements OA and OB . If we complete the parallelogram of which OA and OB are the adjacent sides, we obtain the diagonal OP . If we let x^r be the coordinates of P , it is a simple theorem on projections to prove that when the axes are orthogonal Cartesian



$$x^r = a^r + b^r.$$

Hence this is a vector equation which is true in the fundamental Cartesian system of § 1. Consequently it is true in every rectilinear coordinate system, and we see that *the point of coordinates $(a^r + b^r)$ is that which is given by the vectorial addition of the two displacements OA and OB .*



If we are given three displacements OA , OB , OC , we may complete the parallelepiped whose edges are these displacements and thus obtain the diagonal OP . If the displacements are represented by the three vectors a^r , b^r , c^r , we see that OQ represents the vector $(a^r + b^r + c^r)$ and hence OP represents the sum $(a^r + b^r + c^r)$. Conversely, if we resolve OP into the three displacements OA , OB and OC , we see that the coordinates of P are $(a^r + b^r + c^r)$.

EXAMPLES.

1. Show that the point P whose coordinates are ka^r , where k is a constant, lies on the line OA and that $k = OP/OA$.

[k is an invariant and has the value OP/OA in the fundamental Cartesian system.]

2. If we draw from the point x_0^r a line equal and parallel to OA , show that the resulting point has for its coordinates $(x_0^r + a^r)$.

3. Prove that *equal and parallel displacements are represented by the same contravariant vector*.

4. A , B , C are three given points. If we take three points L , M , N on the lines OA , OB , OC at distances $\alpha \cdot OA$, $\beta \cdot OB$, $\gamma \cdot OC$ respectively from O , show that the point obtained by compounding the displacements OL , OM , ON vectorially has the coordinates $(\alpha a^r + \beta b^r + \gamma c^r)$.

5. Show that the point $\frac{\lambda x_1^r + \mu x_2^r}{\lambda + \mu}$ lies on the line joining the points x_1^r and x_2^r , dividing them in the ratio μ/λ .

3. The unit points and the geometrical interpretation of rectilinear coordinates.

There are three points which are of special interest in the rectilinear system x^r . These are the points whose coordinates are $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. We shall call them the *unit points of the coordinate system* and we shall denote them by E_1 , E_2 , E_3 respectively. If we write the coordinates of the unit point E_i as $e_{(i)}^r$, we see that in every coordinate system

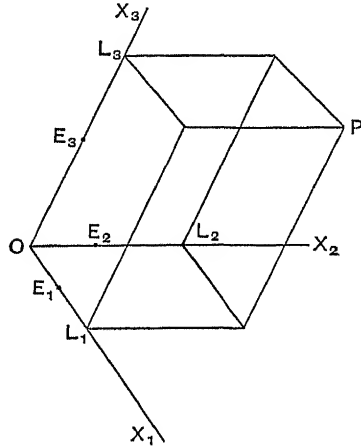
$$e_{(i)}^r = \delta_i^r, \dots \dots \dots (6)$$

where δ_i^r equals zero if $r \neq i$, and equals unity if $r = i$.

Here we enclose the index i in brackets to show that it does not denote tensorial character but merely indicates the particular unit point we are dealing with. We notice that the unit points are three points lying one in each of the coordinate axes. Also, having regard to (6), we have the relations

$$x^r = x^i e_{(i)}^r = x^1 e_{(1)}^r + x^2 e_{(2)}^r + x^3 e_{(3)}^r \dots \dots \dots (7)$$

Let us take the point L_1 whose coordinates are $x^1 e'_{(1)}$. This is a point on the axis OX_1 at such a distance that $OL_1 = x^1 OE_1$. Similarly L_2 and L_3 are points on the other two coordinate axes. If we complete the parallelepiped of which OL_1, OL_2, OL_3 are the edges, we know that the point P arrived at in this way has for its coordinates $x^1 e'_{(1)} + x^2 e'_{(2)} + x^3 e'_{(3)}$. Hence from (7) we see that the coordinates of P are x^r . This gives us a geometrical interpretation of rectilinear coordinates. We draw through P planes parallel to the coordinate planes, which intersect the axes in L_1, L_2, L_3 . Then the three ratios $OL_1/OE_1, OL_2/OE_2, OL_3/OE_3$ are the rectilinear coordinates of P . In other words, to determine a rectilinear coordinate system uniquely we require three axes OX_1, OX_2, OX_3 and three unit points E_1, E_2, E_3 on these axes.



We shall now prove that we can take as unit points of a rectilinear system any three distinct points which are not coplanar with O . Let us return to our original Cartesian system y^r , and let $y'_{(i)}$ be the Cartesian coordinates of the three given points, which we may call E_1, E_2, E_3 . The three points do not all lie on the same plane through O , and hence the quantities $y'_{(i)}$ do not satisfy simultaneously a linear equation of the form (3), p. 36. The necessary and sufficient condition that this be so is that the determinant

$$| y'_{(i)} | \dots \dots \dots (8)$$

be different from zero. Let us make the linear transformation

$$y^r = y'_{(i)} x^i \dots \dots \dots (9)$$

to new variables x^r . Since (8) is not zero, this transformation is reversible and defines a unique rectilinear system x^r . We shall prove that the unit points of the system x^r are the three given points. For the j th unit point has coordinates $e'_{(j)}$ in the x -system, and therefore its Cartesian coordinates are, from (9),

$$y^r = y'_{(i)} e'_{(j)} = y'_{(i)} \delta^i_j = y'_{(j)}.$$

Hence E_1, E_2, E_3 are the unit points of the rectilinear system x^r , which proves our theorem.

We remark here that the system of coordinates x^r is a *Cartesian* system if the unit points are at unit distances from the origin, that is, if

$$OE_1 = OE_2 = OE_3 = 1.$$

These coordinates are *orthogonal* if the coordinate planes are perpendicular to each other, otherwise they are oblique. We conclude that all Cartesian coordinate systems with the point O as origin form a subset of the set of recti-

linear coordinate systems with the same origin. In fact, a rectilinear system is only a Cartesian system modified by introducing scales of measurement along each axis.

EXAMPLES.

1. If the equations $a_s^r x^s = 0$ have two distinct non-zero solutions show that all the cofactors of $|a_s^r|$ are zero.

In the coordinate system x^r let the quantities a_s^r be taken as the components of a tensor. Then $a_s^r x^s = 0$ is a tensor equation which is true in all coordinate systems if it is true in any one system.

Let x_1^r and x_2^r be the two distinct solutions and let us take a new coordinate system \bar{x}^r in which these two points are the unit points \bar{E}_1, \bar{E}_2 . Then we have

$$\begin{aligned}\bar{a}_s^r \bar{x}_1^s &= \bar{a}_s^r \bar{e}_{(1)}^s = \bar{a}_s^r \delta_1^s = \bar{a}_1^r = 0, \\ \bar{a}_s^r \bar{x}_2^s &= \bar{a}_s^r \bar{e}_{(2)}^s = \bar{a}_s^r \delta_2^s = \bar{a}_2^r = 0.\end{aligned}$$

The determinant $|\bar{a}_s^r|$ in the new system is

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{a}_3^1 & \bar{a}_3^2 & \bar{a}_3^3 \end{vmatrix}$$

and we see that all the cofactors are zero in the new system. But the cofactors form a tensor and hence they must vanish in every system. Thus all the cofactors of $|a_s^r|$ are zero.

2. If the equations $a_s^r x^s = b^r$ have two distinct solutions, show that $|a_s^r|$ must be zero, and that if there are three independent solutions then all the cofactors of $|a_s^r|$ must be zero.

[Three independent solutions are three which are such that one is *not* a linear combination of the other two.]

3. If we denote by \overrightarrow{OP} the vector displacement OP , prove that

$$\overrightarrow{OP} = x^r \overrightarrow{OE}_{(r)},$$

where x^r are the coordinates of P and $\overrightarrow{OE}_{(r)}$ are the unit points of the coordinate system.

4. If $\overrightarrow{OE}_{(r)}$ are the unit points of the rectilinear system \bar{x}^r , show that

$$\overrightarrow{OE}_{(r)} = \gamma_r^s \overrightarrow{OE}_{(s)}.$$

[Note that $\overrightarrow{\bar{x}^r \overrightarrow{OE}_{(r)}} = \overrightarrow{OP} = x^r \overrightarrow{OE}_{(r)} = \gamma_r^s x^r \overrightarrow{OE}_{(s)}$, and this relation is true for all values of \bar{x}^r . Thus we see that a covariant vector transforms in the same way (covariantly) as the unit points, and a contravariant vector in the opposite way (contravariantly).]

5. Show that we can always choose an orthogonal Cartesian system such that any given contravariant vector may have two of its components zero in that system.

4. The distance between two points and the fundamental double tensor. The ε -systems.

Let P and Q be two points whose rectilinear coordinates are x_1^r and x_2^r . If y_1^r and y_2^r are their coordinates referred to the fundamental Cartesian system, it is well-known that the distance δ between P and Q is given by

$$\delta^2 = (PQ)^2 = (y_1^1 - y_2^1)^2 + (y_1^2 - y_2^2)^2 + (y_1^3 - y_2^3)^2$$

But, from (2), p. 36,

$$y_1^r = \alpha_m'^r x_1^m, \quad y_2^r = \alpha_m'^r x_2^m.$$

Hence we have

$$\delta^2 = g_{mn} (x_1^m - x_2^m) (x_1^n - x_2^n), \quad (10)$$

where we put

$$g_{mn} = \alpha_m'^1 \alpha_n'^1 + \alpha_m'^2 \alpha_n'^2 + \alpha_m'^3 \alpha_n'^3 = \alpha_m'^r \alpha_n'^r. \quad . . (11)$$

We note that g_{mn} is a symmetric system, that is,

$$g_{mn} = g_{nm}.$$

In particular the distance from O to the point P , whose coordinates are x^r , is given by

$$\delta^2 = g_{mn} x^m x^n. \quad (12)$$

Now δ is an invariant and hence $g_{mn} x^m x^n$ is an invariant. Since g_{mn} is symmetric and x^r is an arbitrary contravariant vector, we can therefore conclude from the quotient law of tensors that g_{mn} is a *double covariant tensor*. We shall refer to it as the *fundamental* or *metric tensor*. We notice incidentally that the quadratic form (12) is positive definite and can vanish only if $x^r = 0$.

If we denote by g the determinant $|g_{mn}|$ and by G^{mn} the cofactor of g_{mn} in g , we know (cf. p. 15) that

$$3! g = e^{rst} e^{mnp} g_{rm} g_{sn} g_{tp}, \quad 2! G^{rs} = e^{rmn} e^{spq} g_{mp} g_{nq}.$$

Since e^{rst} is a relative tensor of weight 1 (p. 30), we see that g is a relative invariant of weight 2 and G^{rs} is a relative tensor of weight 2. Moreover, from (11), we have

$$g = |\alpha_m'^r \alpha_n'^r| = |\alpha_m'^r|^2,$$

and so g is positive and different from zero. Hence if we divide G^{rs} by g and denote the quotient by g^{rs} , we obtain the important result that g^{rs} , the cofactor of g_{rs} divided by g , is an absolute contravariant tensor. Also, if we put

$$\varepsilon_{rst} = \sqrt{g} e_{rst}, \quad \varepsilon^{rst} = \frac{1}{\sqrt{g}} e^{rst}, \quad \dots \quad (13)$$

we have immediately that these quantities are *absolute tensors*, which we shall refer to as *the ε -systems*.

If we are given a contravariant vector A^r , we have connected with it an invariant A defined by the equation

$$A = (g_{mn} A^m A^n)^{\frac{1}{2}}. \quad \dots \quad (14)$$

We call A the *magnitude of the vector A^r* . Similarly we define the magnitude of the covariant vector B_r by the equation

$$B = (g^{mn} B_m B_n)^{\frac{1}{2}}. \quad \dots \quad (15)$$

A *unit vector* is one whose magnitude is unity and hence, if λ^r and μ_r are unit vectors, we have the equations

$$g_{mn} \lambda^m \lambda^n = 1, \quad g^{mn} \mu_m \mu_n = 1. \quad \dots \quad (16)$$

A unit vector is often called a *versor*.

EXAMPLES.

1. Prove that

$$g_{sm} g^{mr} = g^{rn} g_{ms} = \delta_s^r.$$

2. Prove that the distances of the unit points from the origin are given by

$$OE_1 = \sqrt{g_{11}}, \quad OE_2 = \sqrt{g_{22}}, \quad OE_3 = \sqrt{g_{33}}.$$

3. Show that the necessary and sufficient conditions that a coordinate system be Cartesian are $g_{11} = g_{22} = g_{33} = 1$.

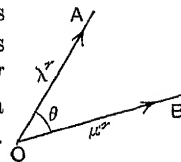
4. Show that a versor λ^r defines a point A at unit distance from the origin and consequently that every versor defines a unique direction.

5. Show that for a rectangular Cartesian system we have

$$g = 1, \quad \varepsilon_{rst} = e_{rst}, \quad \varepsilon^{rst} = e^{rst}, \quad g_{mn} = \begin{Bmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{Bmatrix} = g^{mn}.$$

5. The angle between two directions; orthogonality.

Let A and B be two points at unit distances from the origin, and let λ^r and μ^r be the versors giving the directions OA , OB , that is, λ^r , μ^r are the coordinates of A , B . We wish to find a formula for the angle θ between OA and OB . In the triangle OAB we have



$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cos \theta = 2(1 - \cos \theta) \quad (17)$$

Now

$$\begin{aligned} AB^2 &= g_{mn} (\lambda^m - \mu^m) (\lambda^n - \mu^n) \\ &= g_{mn} \lambda^m \lambda^n + g_{mn} \mu^m \mu^n - 2g_{mn} \lambda^m \mu^n \\ &= 2(1 - g_{mn} \lambda^m \mu^n), \end{aligned}$$

since λ^r and μ^r are unit vectors. It follows from (17) that

$$\cos \theta = g_{mn} \lambda^m \mu^n. \quad (18)$$

If A^r is a contravariant vector defining a point P , then the versor A^r/A , where A is the magnitude of A^r , defines the point on OP at unit distance from the origin. Hence A^r/A is the versor giving the direction of the line joining O to the point A^r . We conclude that, if A^r and B^r are two contravariant vectors whose magnitudes are A and B , then the angle θ between the directions of A^r and B^r is given by

$$\cos \theta = \frac{g_{mn} A^m B^n}{AB}. \quad (19)$$

We can now write down the condition that *two directions may be orthogonal*. If the directions λ^r , μ^r are perpendicular, $\theta = \pi/2$ and we have

$$g_{mn} \lambda^m \mu^n = 0. \quad (20)$$

Similarly the lines joining O to the points A^r and B^r are orthogonal if

$$g_{mn} A^m B^n = 0.$$

EXAMPLES.

1. Prove that the angles between the coordinate axes are given by

$$\cos \theta_{23} = \frac{g_{23}}{\sqrt{g_{22} g_{33}}}, \quad \cos \theta_{31} = \frac{g_{31}}{\sqrt{g_{33} g_{11}}}, \quad \cos \theta_{12} = \frac{g_{12}}{\sqrt{g_{11} g_{22}}}$$

2. If $\alpha_1, \alpha_2, \alpha_3$ are the angles which the versor λ^r makes with the coordinate axes, show that $\cos \alpha_1 = g_{1m} \lambda^m / \sqrt{g_{11}}$, etc.

$$[\sqrt{g_{11}} \cos \alpha_1 = g_{mn} \lambda^m e_{(1)}^n = g_{mn} \lambda^m \delta_1^n = g_{1m} \lambda^m].$$

3. If the coordinates are rectangular Cartesian, show that the cosines of the angles which the versor λ^r makes with the coordinate axes are $\lambda^1, \lambda^2, \lambda^3$. These are usually referred to as *direction cosines*.

4. Prove that in an oblique Cartesian system of coordinates the g_{mn} have the values

$$\begin{cases} 1, & \cos \theta_{12}, & \cos \theta_{13} \\ \cos \theta_{21}, & 1, & \cos \theta_{23} \\ \cos \theta_{31}, & \cos \theta_{32}, & 1 \end{cases}$$

5. If λ^r, μ^r are two unit vectors, show that the angle θ between their directions satisfies the equation

$$\sin^2 \theta = (g_{mn} g_{rs} - g_{mr} g_{ns}) \lambda^m \lambda^n \mu^r \mu^s.$$

6. If $\gamma_1, \gamma_2, \gamma_3$ are the direction cosines of the direction λ^r , show that in oblique Cartesian coordinates they satisfy the relation $g^{mn} \gamma_m \gamma_n = 1$.

6. Associated tensors.

We shall now explain the important processes of raising and lowering the indices of a tensor in order to obtain new tensors.

Let A^r be a given contravariant vector. If we multiply it by the tensor g_{mn} and contract we get the covariant vector $g_{rm} A^m$, which we shall denote by A_r . Thus

$$A_r = g_{rm} A^m, \quad (21)$$

and we have changed the index from a superscript to a subscript. This is the process of *lowering an index*. Let us now take the covariant vector A_r and let us *raise* the index by multiplying it by the tensor g^{mn} and contracting. We have

$$g^{rm} A_m = g^{rm} g_{mn} A^n = \delta_n^r A^n = A^r. \quad (22)$$

Hence if we begin by lowering an index and then raise it again by the processes that we have just explained we see that we get back to the original vector. The processes of raising and lowering indices are therefore completely reciprocal. Two vectors A^r and A_r , related in this way, are called *associated vectors* and are usually denoted by the same letter A . In fact they are so intimately connected that we sometimes say that they represent

the *same* vector, the A^r giving the contravariant components and the A_r the covariant components of that vector.

Let us next consider the double covariant tensor A_{rs} . Here there are two indices that can be raised. If we wish to raise the first index we multiply by g^{rn} and contract with respect to the first index. We shall write this process

$$A^r_{\cdot s} = g^{rn} A_{ns} \dots \dots \dots (23)$$

In order that we may know which index has been raised, we place a dot before the s , showing clearly that s is the second index and that the first has been raised. The process of raising the second index will similarly be written

$$A_r^{\cdot s} = g^{sn} A_{rn} \dots \dots \dots (24)$$

Lastly we can raise both indices in accordance with the formulae

$$A^{rs} = g^{rn} g^{sn} A_{mn} \dots \dots \dots (25)$$

We remark here that if A_{rs} is *symmetrical* it is a simple matter to see that $A^r_{\cdot s}$ and $A_r^{\cdot s}$ have exactly the same values, and both can be written A^r_s , the dot being now unnecessary. The student is left to formulate the rules for lowering indices in a double tensor and to show that if we lower the superscripts in (23), (24) and (25) we return to the original tensor.

These processes can obviously be extended to tensors of any order. We shall illustrate them by one further example. Let A^{rst} be a triple contravariant tensor. The process of lowering the second index is given by

$$A^{r\cdot s t} = g_{sm} A^{rmt} \dots \dots \dots (26)$$

We insert just sufficient dots to show that the index has been brought down vertically.

All tensors obtained from each other in this way are called *associated tensors*.

There are two results which are of interest in this connection.

(i) In any term where there appears a *dummy* index we can raise this index from its lower position and at the same time lower it from its upper position without altering the value of the term. For example, we have

$$A^r_{\cdot s} B_r = g^{rm} A_{ms} g_{rn} B^n = \delta^n_m A_{ms} B^n = A_{ms} B^m = A_{r\cdot s} B^r,$$

and we see that the dummy index r has been raised and lowered in the way mentioned.

(ii) If there is a *free* index in every term of a tensor equation, we obtain an equivalent equation by raising or lowering this free index in every place it occurs. For example, the equations

$$A_{rst} = B_{rs} C_t$$

are equivalent to the equations

$$g^{rm} A_{mst} = g^{rm} B_{ms} C_t,$$

that is, to

$$A^r_{st} = B^r_{s} C_t,$$

in which we see that the index r has been everywhere raised.

EXAMPLES.

1. Show that the magnitude of a vector equals the magnitude of its associate.

2. If A is the magnitude of A^r , then $A^2 = A^r A_r$.

3. If A^r and B^r are two vectors of magnitudes A and B , show that $A^r B_r = AB \cos \theta$, where θ is the angle between the directions of A^r and B^r .

4. If the coordinates are orthogonal Cartesian, show that the components of a vector and its associate are identical.

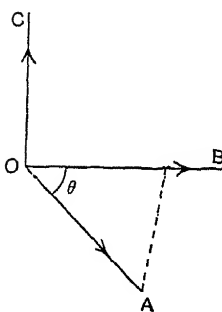
[In ordinary vector analysis there is invariably implied a reference to an orthogonal coordinate system, and this is why contravariant and covariant vectors are not distinguished in that calculus.]

5. Show that g_{mn} , g^{mn} and δ^n_m are associated tensors.

6. Show that e_{rst} and ε^{rst} are associated tensors.

[We have $g e_{rst} = e^{mnp} g_{rm} g_{sn} g_{tp}$, that is, $e_{rst} = e^{mnp} g_{rm} g_{sn} g_{tp}$.]

7. Scalar and vector products of vectors.



Let OA and OB be two displacements given by the vectors A^r , B^r whose magnitudes are A and B . If θ is the angle between these directions we know that

$$g_{mn} A^m B^n = AB \cos \theta. \quad (27)$$

This invariant $g_{mn} A^m B^n$ is called the *scalar product* of A^r and B^r , and we know that it can be written in the equivalent forms

$$g_{mn} A^m B^n = A_m B^m = A^m B_m = g^{mn} A_m B_n,$$

where A_r , B_r are the associated vectors.

If λ^r is the unit vector in the direction of B^r , we therefore have

$$g_{mn} A^m \lambda^n = A_m \lambda^m = A \cos \theta. \dots \dots (28)$$

But $A \cos \theta$ is just the projection of OA on the line OB . Hence the projection of A^r in the direction λ^r is given by $A_m \lambda^m$.

Let us next consider the vector

$$C^r = \varepsilon^{rmn} A_m B_n. \dots \dots (29)$$

To find the geometrical interpretation of this vector, let us choose a special coordinate system as follows. We take the \bar{x}^1 -axis along OA and the \bar{x}^2 -axis perpendicular to OA and in the plane of OA and OB ; finally we take the \bar{x}^3 -axis perpendicular to both OA and OB , so that our new system is *orthogonal and Cartesian*. In this special

system we have
 $\bar{A}^r \equiv (A, 0, 0), \quad \bar{B}^r \equiv (B \cos \theta, B \sin \theta, 0),$
 and the associated vectors have the same components. Also

$$\bar{e}^{rmn} = e^{rmn}.$$

Hence the components of \bar{C}^r in the new system are

$$\bar{C}^r = \bar{e}^{rmn} \bar{A}_m \bar{B}_n = e^{r1n} A \bar{B}. \quad e^{r12} A B \sin \theta,$$

that is,

$$\bar{C}^1 = \bar{C}^2 = 0, \quad \bar{C}^3 = A B \sin \theta.$$

In other words the vector \bar{C}^r lies along $O\bar{X}_3$, the perpendicular to the plane of OA and OB , and its magnitude is $AB \sin \theta$. These are purely geometrical relations and are therefore independent of any particular coordinate system. We may then return to the original coordinate system and say that C^r has its direction along the perpendicular to the plane of OA and OB and its magnitude is $AB \sin \theta$.

Although we have proved that C^r lies along the perpendicular to the plane OAB , we have still to decide what is its sense along this line. Multiplying (29) by C_r and summing with respect to r from 1 to 3, we get

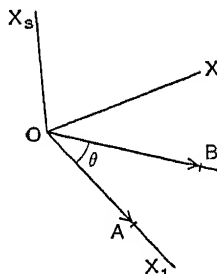
$$C^2 = \varepsilon^{rmn} C_r A_m B_n = \varepsilon_{mnp} A^m B^n C^p,$$

that is, C^r is in such a direction that the invariant $\varepsilon_{mnp} A^m B^n C^p$ is positive.

We are therefore led to examine the sign of the expression

$$\varepsilon_{mnp} A^m B^n C^p, \dots \dots (30)$$

where A^r, B^r, C^r are any three vectors. This quantity is an invariant and if we take the same special Cartesian system as before we see that it equals $\bar{C}^3 A B \sin \theta$. It will therefore vanish only if $\bar{C}^3 = 0$, which means that C^r lies in the plane OAB . Consequently (30) vanishes only if one vector becomes coplanar with the other two. If then we deform the triad (A^r, B^r, C^r) continuously in such a way that they never become coplanar we see that (30) varies continuously and must always retain the same sign since it cannot vanish. Let us deform it continuously until A^r coincides with the positive x^1 -axis, B^r with the positive x^2 -axis and C^r with the x^3 -axis. We say that the original triad has a positive or negative



orientation according as the deformed C^r lies along the positive or negative direction of the x^3 -axis. Now (30) becomes for the deformed triad $+ABC\sqrt{g}$ or $-ABC\sqrt{g}$ according as we are in the first or second case. We conclude that the triad (A^r, B^r, C^r) has a positive or negative orientation according as (30) is positive or negative.

The vector C^r , defined by the equations (29), is therefore in such a direction that the orientation of the triad (A^r, B^r, C^r) is positive. The vector (29) is called *the vector product of A^r, B^r* . If we denote by ν^r the versor of the perpendicular to A^r, B^r in the correct sense we can write

$$\varepsilon^{r mn} A_m B_n = AB \sin \theta \nu^r. \quad (31)$$

EXAMPLES.

1. If α, β, γ are the angles between each pair of the triad of directions OA, OB, OC , prove that

$$\begin{array}{ccccccc} 1, & \cos \gamma, & \cos \beta & & & & \\ \cos \gamma, & 1, & \cos \alpha & \sin^2 \gamma \sin^2 \varphi, & & & \\ \cos \beta, & \cos \alpha, & 1 & & & & \end{array}$$

where φ is the angle which OC makes with the plane OAB .

Let A^r, B^r, C^r be three vectors along OA, OB, OC and let their magnitudes be A, B, C . If ν^r is the unit vector orthogonal to A^r, B^r we have the angle between C^r and ν^r equal to $\frac{\pi}{2} - \varphi$, that is,

$$C \sin \varphi = C_r \nu^r.$$

Hence we conclude from (31) that

$$ABC \sin \gamma \sin \varphi = \varepsilon^{r mn} C_r A_m B_n = \varepsilon_{m n p} A^m B^n C^p.$$

Now this last expression is an invariant, and if we take the coordinates to be Cartesian and such that OA, OB, OC are the axes we have

$$\varepsilon_{m n p} A^m B^n C^p = \sqrt{g} e_{123} ABC = \sqrt{g} ABC.$$

Moreover in this system α, β, γ are the angles between the axes, and g_{mn} have the values

$$\begin{Bmatrix} 1, & \cos \gamma, & \cos \beta \\ \cos \gamma, & 1, & \cos \alpha \\ \cos \beta, & \cos \alpha, & 1 \end{Bmatrix}.$$

Therefore combining the two preceding equations we have the result required.

2. Prove that the projections of the vector A^r on the coordinate axes are $A_1/\sqrt{g_{11}}, A_2/\sqrt{g_{22}}, A_3/\sqrt{g_{33}}$. Hence if the coordinates are Cartesian these projections are A_1, A_2, A_3 .

3. Show that the unit vector perpendicular to the plane OE_2E_3 has for its covariant components

$$\sqrt{g}e_{r23}/\sqrt{g_{22}g_{33}}\sin\theta_{23}.$$

[The covariant components are given by

$$\varepsilon_{rmn}e_{(2)}^me_{(3)}^n/OE_2\cdot OE_3\sin\theta_{23}].$$

4. If λ^r, μ^r, ν^r are three mutually orthogonal unit vectors, prove that

$$\varepsilon_{rst}\lambda^r\mu^s\nu^t = \pm 1.$$

5. If A^r, B^r are two vectors and θ the angle between them, then

$$A^2B^2\sin^2\theta = \delta_{rs}^{mn}A^rB^sA_mB_n.$$

[We may either choose a special Cartesian system of coordinates or use (31).]

6. If we put $C^{rs} = A^rB^s - A^sB^r$, show that C^{rs} is skew-symmetric and that $\frac{1}{2}\varepsilon_{rmn}C^{mn}$ are the covariant components of the vector product of A^r, B^r .

8. Areas and volumes.

Let P_1 and P_2 be two given points; we wish to find an expression for the area of the triangle OP_1P_2 . Let $x_{(1)}^r, x_{(2)}^r$ be the coordinates of P_1 and P_2 . We know that the magnitude of the vector $\varepsilon_{rmn}x_{(1)}^mx_{(2)}^n$ is $OP_1 \cdot OP_2 \sin(P_1OP_2)$. But this is twice the area of the triangle OP_1P_2 . Denoting this area by Δ , we have therefore

$$4\Delta^2 = g^{mn}\varepsilon_{mnpq}x_{(1)}^px_{(2)}^q\varepsilon_{rst}x_{(1)}^rx_{(2)}^s \quad (32)$$

which gives an expression for the area Δ . Moreover if ν^r is the unit vector orthogonal to the plane of the triangle and chosen such that the orientation of the triad (OP_1, OP_2, ν^r) is positive, then equations (31), p. 48, give

$$\varepsilon_{rmn}x_{(1)}^mx_{(2)}^n = 2\Delta\nu_r. \quad (33)$$

Let P_3 be a third point whose coordinates are $x_{(3)}^r$. As in Ex. 1 (p. 48) the invariant $\varepsilon_{mnp}x_{(1)}^mx_{(2)}^nx_{(3)}^p$ has for its value

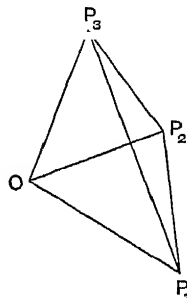
$$\varepsilon_{mnp}x_{(1)}^mx_{(2)}^nx_{(3)}^p = OP_1 \cdot OP_2 \cdot OP_3 \sin(P_1OP_2)\sin\varphi, \quad (34)$$

where φ is the angle which OP_3 makes with the plane OP_1P_2 . Now $OP_1 \cdot OP_2 \sin(P_1OP_2)$ is twice the area of the triangle OP_1P_2 and $OP_3 \sin\varphi$ is the perpendicular from P_3 on the plane OP_1P_2 . Hence the right-hand side of (34) is six times the volume of the tetrahedron $OP_1P_2P_3$. Denoting this volume by V , we have

$$6V = \varepsilon_{mnp}x_{(1)}^mx_{(2)}^nx_{(3)}^p \quad (35)$$

Since $\varepsilon_{mnp} = \sqrt{g}e_{mnp}$, this equation may be written

(E 287)



$$6V = \sqrt{g} \begin{vmatrix} x_{(1)}^1 & x_{(1)}^2 & x_{(1)}^3 \\ x_{(2)}^1 & x_{(2)}^2 & x_{(2)}^3 \\ x_{(3)}^1 & & \end{vmatrix} \quad (36)$$

EXAMPLES.

1. Prove that the area of the triangle $OE_2 E_3$ is $\frac{1}{2} (g^{11})^{\frac{1}{2}}$.
2. Prove that the volume of the tetrahedron $OE_1 E_2 E_3$ is $\frac{1}{6} \sqrt{g}$.
3. If a, b, c are the edges OA, OB, OC of a tetrahedron and α, β, γ are the angles between these edges, show that

$$6V = abc \begin{vmatrix} 1, & \cos \gamma, & \cos \beta \\ \cos \gamma, & 1, & \cos \alpha \\ \cos \beta, & \cos \alpha, & 1 \end{vmatrix}$$

[Calculate the invariant (35) in a Cartesian system in which OA, OB, OC are the axes.]

EXAMPLES III.

1. If the coordinate system is orthogonal, prove that

$$g^{11} = \frac{1}{g_{11}}, \quad g^{22} = \frac{1}{g_{22}}, \quad g^{33} = \frac{1}{g_{33}}$$

2. If $g_{mn} A^m A^n = 0$, then $A^r = 0$.
3. If θ_{23} is the angle between OX_2 and OX_3 , show that

$$\sin^2 \theta_{23} = \frac{g^{11}}{g_{22} g_{33}}$$

4. If λ^r is a unit vector, prove that the cosines of the angles which this direction makes with the coordinate axes are

$$\frac{\lambda_1}{\sqrt{g_{11}}}, \quad \frac{\lambda_2}{\sqrt{g_{22}}}, \quad \frac{\lambda_3}{\sqrt{g_{33}}}$$

5. Show that

$$1 - \cos^2 \theta_{23} - \cos^2 \theta_{31} - \cos^2 \theta_{12} + 2 \cos \theta_{12} \cos \theta_{23} \cos \theta_{31} = \frac{g}{g_{11} g_{22} g_{33}}$$

6. If A^{rs} is a skew-symmetric contravariant tensor, then

$$(\sqrt{g} A^{23}, \sqrt{g} A^{31}, \sqrt{g} A^{12})$$

are the components of a covariant vector.

[It is the vector $\frac{1}{2} \epsilon_{r mn} A^{mn}$.]

7. If A_{rs} is a skew-symmetric covariant tensor, then

$$(A_{23}/\sqrt{g}, A_{31}/\sqrt{g}, A_{12}/\sqrt{g})$$

are the components of a contravariant vector.

[These two results prove that we can always turn a skew-symmetric double tensor into a vector.]

8. If A_{rs} is skew-symmetric and $C^r = \varepsilon^{rmn} A_{mn}$, prove that $2 A_{rs} = \varepsilon_{rst} C^t$.

9. If $x'_{(1)}$ and $x'_{(2)}$ are the coordinates of two points P_1 and P_2 and if

$$A^{rs} = \begin{vmatrix} x'_{(1)} & x'_{(2)} \\ x^s_{(1)} & x^s_{(2)} \end{vmatrix},$$

prove that the area Δ of the triangle OP_1P_2 is given by

$$8\Delta^2 = A^{rs} A_{rs} = g_{rm} g_{sn} A^{rs} A^{mn}.$$

10. If the coordinates are orthogonal Cartesian show that A^{23} , A^{31} , A^{12} of Ex. 9 are twice the areas of the triangles projected on the co-ordinate planes. Deduce from Ex. 9 that

$$4\Delta^2 = (A^{23})^2 + (A^{31})^2 + (A^{12})^2.$$

11. If

$$A^{rst} = \begin{vmatrix} x'_{(1)} & x^s_{(1)} & x^t_{(1)} \\ x'_{(2)} & x^s_{(2)} & x^t_{(2)} \\ x'_{(3)} & x^s_{(3)} & x^t_{(3)} \end{vmatrix},$$

where $x'_{(i)}$ are the coordinates of three points P_1, P_2, P_3 . Prove that

$$(i) \quad A^{rst} = \sqrt{g} A^{123} \varepsilon^{rst}, \quad (ii) \quad A^{rst} = 6V \varepsilon^{rst},$$

V being the volume of $OP_1P_2P_3$.

12. If we put

$$B^{rs} = \begin{vmatrix} x'_{(1)} & x'_{(2)} & x'_{(3)} \\ x^s_{(1)} & x^s_{(2)} & x^s_{(3)} \\ 1 & 1 & 1 \end{vmatrix},$$

show that the area of the triangle $P_1P_2P_3$ is given by

$$8\Delta^2 = B^{rs} B_{rs} = g_{rm} g_{sn} B^{rs} B^{mn}.$$

13. If

$$B^{rst} = \begin{vmatrix} x'_{(1)} & x'_{(2)} & x'_{(3)} & x'_{(4)} \\ x^s_{(1)} & x^s_{(2)} & x^s_{(3)} & x^s_{(4)} \\ x^t_{(1)} & x^t_{(2)} & x^t_{(3)} & x^t_{(4)} \\ 1 & 1 & 1 & 1 \end{vmatrix},$$

where $x'_{(1)}, x'_{(2)}, x'_{(3)}, x'_{(4)}$ are the coordinates of four points $P_1P_2P_3P_4$, and if V is the volume of the tetrahedron $P_1P_2P_3P_4$, then

$$B^{rst} = 6V \varepsilon^{rst}.$$

[Prove that B^{rst} is completely skew-symmetric and that

$$\sqrt{g} B^{123} = 6V.]$$

14. If we are given N points $x'_{(\alpha)}$, ($\alpha = 1, 2, 3, \dots, N$), show that the mean centre of these points for multiples $\lambda_{(\alpha)}$ has the coordinates

$$\xi^r = \frac{\sum_{\alpha=1}^N \lambda_{(\alpha)} x_{(\alpha)}^r}{\left\{ \sum_{\alpha=1}^N \lambda_{(\alpha)} \right\}}.$$

[This is a vector equation which is easily proved to be true in orthogonal Cartesian coordinates.]

15. Show that the four points

$$\frac{x_1^r + \lambda_1 x_2^r}{1 + \lambda_1}, \quad \frac{x_1^r + \lambda_2 x_2^r}{1 + \lambda_2}, \quad \frac{x_1^r + \lambda_3 x_2^r}{1 + \lambda_3}, \quad \frac{x_1^r + \lambda_4 x_2^r}{1 + \lambda_4},$$

all lie on the line joining x_1^r and x_2^r and that their cross-ratio is

$$\frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3} \bigg/ \frac{\lambda_1 - \lambda_4}{\lambda_2 - \lambda_4}.$$

16. If $x_{(i)}^r, x_{(i')}^r$ are the coordinates of 6 points $P_i, P_{i'}$ and if V, V' are the volumes of the tetrahedra $OP_1 P_2 P_3, OP'_1 P'_2 P'_3$ respectively, then

$$36 V V' = \begin{Bmatrix} \prod_{i=1}^3 OP_i \cdot OP_{i'} \\ \cos(11) & \cos(12) & \cos(13) \\ \cos(21) & \cos(22) & \cos(23) \\ \cos(31) & \cos(32) & \cos(33) \end{Bmatrix}$$

where (ij) is the angle between OP_i and $OP_{j'}$.

$$[6V = \sqrt{g} |x_{(i)}^r|, \quad 6V' = \sqrt{g} |x_{(i')}^r|. \quad \text{Hence } 36 V V' = |g_{mn} x_{(i)}^m x_{(i')}^n|.]$$

17. We are given three vectors $\lambda_{(i)}^r$ at the point O . If $\lambda_{(i)}^{(i)}$ is the cofactor of $\lambda_{(i)}^r$ in the determinant $|\lambda_{(i)}^r|$, divided by the determinant, prove that each of the vectors $g^{rs} \lambda_{(i)}^{(s)}$ is orthogonal to two of the vectors $\lambda_{(i)}^r$.

[This result follows from the fact that $\lambda_{(i)}^r \lambda_{(i')}^{(r)} = \delta_i^{i'}$. For example, the vector associate to $\lambda_{(i)}^{(3)}$ is perpendicular to both $\lambda_{(1)}^r$ and $\lambda_{(2)}^r$.]

18. If the axes are Cartesian and $\alpha_1, \alpha_2, \alpha_3$ are the direction cosines of a direction λ^r , show that

$$\begin{aligned} \Sigma \sin^2 \theta_{23} \alpha_1^2 - 2 \Sigma (\cos \theta_{23} - \cos \theta_{12} \cos \theta_{13}) \alpha_2 \alpha_3 &= 1 - \Sigma \cos^2 \theta_{23} \\ &+ 2 \cos \theta_{23} \cos \theta_{31} \cos \theta_{12}. \end{aligned}$$

[Here $\alpha_r = \lambda_r$ and the equation is simply $g^{mn} \lambda_m \lambda_n = 1$ written out in full.]

CHAPTER IV

THE PLANE

1. The equation of a plane.

Let P_1, P_2, P_3 be three points whose coordinates are $x_{(1)}^r, x_{(2)}^r, x_{(3)}^r$. If P is the mean centre of the points P_1, P_2, P_3 for multiples λ, μ, ν , we know from elementary considerations that P lies in the plane of the three given points and we can include

every point of the plane $P_1 P_2 P_3$ by varying λ, μ, ν . Now the equation

$$x^r = \frac{\lambda x_{(1)}^r + \mu x_{(2)}^r + \nu x_{(3)}^r}{\lambda + \mu + \nu} \quad \dots \dots (1)$$

where x^r are the coordinates of P , is a vector equation which is true in any orthogonal Cartesian system of coordinates. It is therefore true in every rectilinear system and gives the coordinates of every point of the plane $P_1 P_2 P_3$. Hence we may regard (1) as one form in which the equation of the plane may be expressed.

The relations (1) can be written

$$\lambda (x^r - x_{(1)}^r) + \mu (x^r - x_{(2)}^r) + \nu (x^r - x_{(3)}^r) = 0.$$

Eliminating λ, μ and ν , we have

$$x^r - x_{(1)}^r = 0,$$

and when this is expanded we get a linear equation in the x 's. Conversely, if x^r satisfies a linear equation, we can take the same steps in the reverse order leading back to the equations (1). *Therefore every linear equation in x^r is the equation of a plane.* Every linear equation is of the form

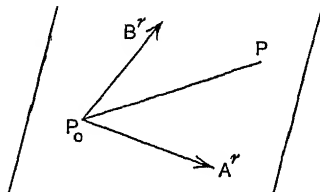
$$a_r x^r = b, \quad \dots \dots (2)$$

where a_r is a covariant vector and b is an invariant, and consequently the equation of a plane can be written in the form (2). We note that if b equals zero the plane passes through the origin whilst if b is not zero the origin does not lie in the plane.

We have now two forms in which the equation of a plane may be written, namely, (1) and (2). There is a third form which is really a modification of (1). Let

P_0 , coordinates x_0^r , be a point of the plane and let A^r, B^r be two vectors at P_0 whose directions lie in the plane. Then the points

$$x_0^r + (\lambda + \mu + \nu) A^r$$



and $x_0^r + (\lambda + \mu + \nu) B^r$ obviously lie in the plane. If we take x_0^r and these two points as the three points of (1), we see that any point of the plane can be expressed in the form

$$x^r = x_0^r + \mu A^r + \nu B^r, \quad \dots \dots (3)$$

for suitable values of μ and ν . This form expresses the coordinates of a point of the plane in terms of a given point of the plane and two directions in it. In particular, any point of a plane passing through the origin can be expressed in the form

$$x^r = \mu A^r + \nu B^r. \quad (4)$$

In other words, all vectors at the origin which are coplanar with the two vectors A^r and B^r are of the form (4).

EXAMPLES.

1. The intercepts of the plane (2) on the axes are of lengths

$$\frac{\sqrt{g_{11}} b}{a_1}, \quad \frac{\sqrt{g_{22}} b}{a_2}, \quad \frac{\sqrt{g_{33}} b}{a_3}.$$

2. If A^r , B^r are unit orthogonal vectors of the plane at the point P_0 , then any point of the plane is given by

$$x^r = x_0^r + r \cos \theta A^r + r \sin \theta B^r,$$

where $r = PP_0$ and θ is the angle which PP_0 makes with A^r .

3. In (1) show that $\lambda : \mu : \nu = \Delta(P P_2 P_3) : \Delta(P P_3 P_1) : \Delta(P P_1 P_2)$.

4. If $x_{(1)}^r$ and $x_{(2)}^r$ are two points not lying in the plane (2), find the ratio $\lambda : \mu$ such that the point $(\lambda x_{(1)}^r + \mu x_{(2)}^r) / (\lambda + \mu)$ should lie in the plane. Deduce that the plane divides space into two parts such that $(a_r x^r - b)$ is positive for the points on one side of the plane and negative for the points on the other side.

2. The perpendicular distance from a point to a plane.

Let us be given a plane whose equation is

$$a_r x^r = b.$$

If $x_{(1)}^r$ and $x_{(2)}^r$ are any two points of the plane then $x_{(1)}^r$ and $x_{(2)}^r$ must satisfy this equation. By subtraction we have

$$a_r (x_{(2)}^r - x_{(1)}^r) = 0. \quad (5)$$

Now $(x_{(2)}^r - x_{(1)}^r)$ is any vector lying in the plane. Hence the relation (5) tells us that the vector

$$a^r = g^{rs} a_s$$

is perpendicular to every vector lying in the plane, and consequently it must be orthogonal to the plane itself. If we denote the magnitude of a_r by a , that is

$$a = \sqrt{g^{mn} a_m a_n}, \quad (6)$$

we conclude that the unit vector perpendicular to the plane is a^r/a .

Let x_0^r be a point which does not lie in the plane. We wish to find the perpendicular distance from x_0^r to the plane. If x^r is any point of the plane, then obviously the perpendicular distance δ is the projection of the vector $(x^r - x_0^r)$ on the perpendicular to the plane. Hence

$$\delta = \frac{a_s (x^s - x_0^s)}{a}.$$

But x^r lies on the plane, that is,

$$a_s x^s = b.$$

Therefore

$$\delta = \frac{b - a_s x_0^s}{a} = \frac{b - a_s x_0^s}{\sqrt{g^{mn} a_m a_n}}. \quad \dots \quad (7)$$

In particular, the perpendicular from the origin on the plane is

$$\delta_0 = \frac{b}{\sqrt{g^{mn} a_m a_n}}. \quad \dots \quad (8)$$

We have seen that the expression $(b - a_s x_0^s)$ has different signs according to the side of the plane on which x_0^r lies. The perpendicular is therefore positive from points on one side of the plane and negative from points on the other side.

We can place the equation of the plane in a special form which is called the *normal form* of the equation. Let us divide equation (2) by a , the magnitude of a_r . The equation of the plane is now of the form

$$\lambda_r x^r = p, \quad \dots \quad (9)$$

where λ_r is a unit vector. Also in the normal form the perpendicular from x_0^r is

$$\delta = p - \lambda_r x_0^r, \quad \dots \quad (10)$$

and p is the perpendicular distance from the origin. In this case the unit vector normal to the plane is $\lambda^r = g^{rs} \lambda_s$.

EXAMPLES.

1. Prove that the angle θ between the planes $a_r x^r = b$ and $a'_r x^r = b'$, is given by $\cos \theta = g^{mn} a_m a'_n / aa'$.

The unit vectors normal to the two planes are a^r/a and a'^r/a' , where a, a' are the magnitudes of a_r and a'_r , and the angle θ between the planes is the same as the angle between the normal vectors. Hence

$$\cos \theta = \frac{g_{mn} a^m a'^n}{a a'}.$$

But $a^r = g^{rm} a_m$ and $a'^r = g^{rm} a'_m$. Therefore $g_{mn} a^m a'^n = g^{rs} a_r a'_s$ and the equation for $\cos \theta$ becomes

$$\cos \theta = \frac{g^{mn} a_m a'_n}{a a'}.$$

2. Prove that the perpendiculars from the unit points E_1, E_2, E_3 on the plane have the lengths $\frac{a_1}{a}, \frac{b-a_2}{a}, \frac{b}{a}$ respectively.

3. Prove that the cosines of the angles, which the perpendicular vector from O on the plane makes with the coordinate axes, are

$$\frac{a_1}{a \sqrt{g_{11}}}, \quad \frac{a_2}{a \sqrt{g_{22}}}, \quad \frac{a_3}{a \sqrt{g_{33}}}.$$

4. Prove that two planes $a_r x^r = b$ and $a'_r x^r = b'$ are orthogonal if $g^{mn} a_m a'_n = 0$, and are parallel if $a'_r = k a_r$.

5. If P is any point on the perpendicular from P_0 on the plane show that $x^r = x_0^r + \varrho a^r/a$, where ϱ is the distance PP_0 . Deduce the length of the perpendicular from P_0 on the plane.

3. The intersection of two planes.

Let

$$\begin{aligned} a_r x^r &= b \\ a'_r x^r &= b' \end{aligned} \quad \dots \dots \dots (11)$$

be the equations of the two given planes. We shall use the notation

$$A_{rs} = \begin{vmatrix} a_r & a'_r \\ a_s & a'_s \end{vmatrix} = a_r a'_s - a_s a'_r, \quad \dots \dots \dots (12)$$

and we see that A_{rs} is a skew-symmetric tensor.

There are only two possibilities: either the planes are parallel or they are not parallel, the case of coincidence being included under that of parallelism.

I. *The planes are parallel.* In this case we must have

$$a'_r = k a_r$$

where k is an invariant. In other words,

$$A_{rs} = 0 \quad \dots \dots \dots (13)$$

If, in addition, the two planes are coincident we see that $k = b'/b$ and we must also have the equations

$$\frac{a'_r}{b'} = \frac{a_r}{b}.$$

These last equations can be written in another form, namely, the equations

$$a_r = \theta_r b, \quad a'_r = \theta_r b' \quad \dots \dots \dots (14)$$

must be satisfied simultaneously.

II. *The planes are not parallel.* The conditions for this are that the quantities A_{rs} are not all zero. This we may write briefly

$$A_{rs} \neq 0, \quad \dots \dots \dots (15)$$

remarking that this does not mean that *all* the A 's are different from zero; it merely expresses that A_{rs} is not a zero tensor. The planes now intersect in a definite straight line. This straight line is obviously in both planes and is perpendicular to both a^r and a'^r . Consequently its direction is that of the vector $\varepsilon^{rmn} a_m a'_n$. If x'_0 is any point on the line of intersection then every other point of the line is given by

$$x^r = x'_0 + t \varepsilon^{rmn} a_m a'_n$$

for all values of t .

We may therefore classify our results as follows:

(A) $A_{rs} \neq 0$, there is a finite line of intersection.

(B) $A_{rs} = 0$,

(i) $\frac{a'_r}{b'} \neq \frac{a_r}{b}$, the planes are parallel and there is a line of intersection at infinity.

(ii) $\frac{a'_r}{b'} = \frac{a_r}{b}$, the planes are coincident.

Let us next consider the equation

$$\lambda (a_r x^r - b) + \mu (a'_r x^r - b') = 0. \quad \dots \dots (16)$$

If the planes (11) are parallel or coincident, (16) is the equation of a parallel or coincident plane for different values of the ratio λ/μ .

If the planes are not parallel, then (16) is a plane passing through their line of intersection. Hence if we vary λ/μ we obtain all the planes passing through the line of intersection of the planes (11). We shall now interpret the ratio λ/μ . Let the plane (16) make angles α, α' with the planes (11). Then, if δ, δ' are the perpendiculars on the two planes (11) from any point on (16), we see that

$$\delta'/\delta = \sin \alpha'/\sin \alpha.$$

But

$$\delta = \frac{b - a_r x^r}{a}, \quad \delta' = \frac{b' - a'_r x^r}{a'}.$$

Therefore, from (16),

$$\lambda a \delta + \mu a' \delta' = 0,$$

and

$$\frac{\lambda}{\mu} = - \frac{a' \delta'}{a \delta} = - \frac{a' \sin \alpha'}{a \sin \alpha}. \quad \dots \dots (17)$$

EXAMPLES.

1. Prove that $\varepsilon^{rmn} A_{mn} = 2 \varepsilon^{rmn} a_m a'_n$.
2. Show that the unit vector drawn parallel to the line of intersection of the planes (11) is $\varepsilon^{rmn} a_m a'_n / a a' \sin \theta$, where θ is the angle between the planes.

3. Show that the angle θ between the planes is given by

$$2 a^2 a'^2 \sin^2 \theta = A_{mn} A^{mn}.$$

4. The point where the line of intersection of the planes cuts the coordinate plane $x^3 = 0$ has the coordinates

$$\left(\frac{b a'_3 - b' a_3}{A_{12}}, \frac{b a'_1 - b' a_1}{A_{21}}, 0 \right).$$

5. By equating the magnitudes of the two equal vectors

$$\varepsilon^{r mn} a_m (\lambda a_n + \mu a'_n) = \mu \varepsilon^{r mn} a_m a'_n,$$

prove that

$$\mu a' \sin \theta = \sin \alpha (\lambda a^2 + \mu a'^2 + 2 \lambda \mu a a' \cos \theta)^{\frac{1}{2}}.$$

4. The intersection of three planes.

Let the equations of the three planes be

$$a_r^{(1)} x^r = b^{(1)}$$

$$a_r^{(2)} x^r = b^{(2)}$$

$$a_r^{(3)} x^r = b^{(3)}$$

or, as they may be more briefly written,

$$a_r^{(i)} x^r = b^{(i)}. \quad (18)$$

We shall use the following notation

$$A_{rst} = \begin{vmatrix} a_r^{(1)} & a_s^{(1)} & a_t^{(1)} \\ a_r^{(2)} & a_s^{(2)} & a_t^{(2)} \\ a_r^{(3)} & a_s^{(3)} & a_t^{(3)} \end{vmatrix} = e_{ijk} a_r^{(i)} a_s^{(j)} a_t^{(k)}, \quad (19)$$

$$A_{rs}^{(1)} = \begin{vmatrix} a_r^{(2)} & a_s^{(2)} \\ a_r^{(3)} & a_s^{(3)} \end{vmatrix} \quad A_{rs}^{(2)} = \begin{vmatrix} a_r^{(3)} & a_s^{(3)} \\ a_r^{(1)} & a_s^{(1)} \end{vmatrix} \quad A_{rs}^{(3)} = \begin{vmatrix} a_r^{(1)} & a_s^{(1)} \\ a_r^{(2)} & a_s^{(2)} \end{vmatrix} \quad (20)$$

These quantities are all skew-symmetric tensors. We shall divide the discussion of the mutual relations of the three planes into the following parts:

I. *All three planes are parallel*, coincidence being considered a special case of parallelism. Reference to the last section will show that we must have

$$A_{rs}^{(i)} = 0,$$

and this of course entails the vanishing of A_{rst} . We divide this case into three sub-cases.

(α) *All three planes are coincident*. The conditions for this are

$$\frac{a_r^{(1)}}{b^{(1)}} = \frac{a_r^{(2)}}{b^{(2)}} = \frac{a_r^{(3)}}{b^{(3)}} = \theta_r,$$

that is,

$$a_r^{(i)} = \theta_r b^{(i)}. \quad (21)$$

(β) *Only two planes are coincident.* Let the last two be the two coincident planes. We must have in this case

$$a_r^{(2)} = \theta_r b^{(2)}, \quad a_r^{(3)} = \theta_r b^{(3)}, \quad a_r^{(1)} \neq \theta_r b^{(1)}.$$

The intersection is a straight line at infinity.

(γ) *No two planes are coincident.* Here no two of the three sets of equations (21) can be satisfied simultaneously. The intersection is a straight line at infinity.

II. *Only two planes are parallel.* Let the last two be parallel. Then we must have (p. 56)

$$A_{rs}^{(1)} = 0, \quad A_{rs}^{(2)} \neq 0, \quad A_{rs}^{(3)} \neq 0.$$

It is a consequence of the vanishing of $A_{rs}^{(1)}$ that

$$A_{rst} = 0.$$

(α) *The two parallel planes are coincident.* For coincidence we must have

$$\frac{a_r^{(2)}}{b^{(2)}} = \frac{a_r^{(3)}}{b^{(3)}} = \theta_r.$$

Also it is a simple matter to verify that under these circumstances

$$\sum_s b^{(i)} A_{rs}^{(i)} = b^{(i)} A_{rs}^{(i)} = 0.$$

The intersection is a finite straight line, namely the intersection of the third plane with the two coincident planes.

(β) *The two parallel planes are not coincident, that is,*

$$\frac{a_r^{(2)}}{b^{(2)}} \neq \frac{a_r^{(3)}}{b^{(3)}}.$$

It follows that

$$b^{(i)} A_{rs}^{(i)} \neq 0.$$

The intersection is now the point common to two parallel straight lines, and so is a point at infinity.

III. *No two planes are parallel.* We must have

$$A_{rs}^{(1)} \neq 0, \quad A_{rs}^{(2)} \neq 0, \quad A_{rs}^{(3)} \neq 0.$$

Now the line of intersection of two of the planes may or may not be parallel to the third. The plane

$$\lambda (a_r^{(2)} x^r - b^{(2)}) + \mu (a_r^{(3)} x^r - b^{(3)}) = 0$$

is a plane passing through the line of intersection of the last two planes. Hence we can or cannot find values of λ, μ such that

$$a_r^{(1)} = \lambda a_r^{(2)} + \mu a_r^{(3)},$$

according as the line of intersection of the last two planes is or is not parallel to the first.

(α) *If the line of intersection of two is not parallel to the remaining plane, we can therefore conclude that*

$$A_{rst} \neq 0,$$

and the intersection is a unique finite point.

(β) If the line of intersection of two is parallel to the remaining, we must have

$$A_{rst} = 0.$$

There are now two possibilities. Either this line of intersection does or does not lie in the remaining plane. The former happens if $b^{(1)} = \lambda b^{(2)} + \mu b^{(3)}$, and this entails

$$b^{(i)} A_{rs}^{(i)} = 0;$$

so that when this is true the intersection is a finite line. In the latter case $b^{(1)} \neq \lambda b^{(2)} + \mu b^{(3)}$, that is,

$$b^{(i)} A_{rs}^{(i)} \neq 0,$$

and the intersection is a point at infinity.

We have now exhausted all the possibilities, and we shall classify our results as follows:

- (A) $A_{rst} \neq 0$, there is a unique finite point of intersection.
- (B) $A_{rst} = 0$, $A_{rs}^{(i)} \neq 0$.
 - (i) $b^{(i)} A_{rs}^{(i)} \neq 0$. There is a unique point of intersection at infinity.
 - (ii) $b^{(i)} A_{rs}^{(i)} = 0$. There is a finite line of intersection.
- (C) $A_{rst} = 0$, $A_{rs}^{(i)} = 0$.
 - (i) $a_r^{(i)} - \theta_r b^{(i)} \neq 0$. There is a unique line of intersection at infinity.
 - (ii) $a_r^{(i)} - \theta_r b^{(i)} = 0$. The three planes are coincident.

[It is to be remarked that the formula $A_{rs}^{(i)} \neq 0$ means that the three tensors $A_{rs}^{(1)}$, $A_{rs}^{(2)}$, $A_{rs}^{(3)}$ are *not all three* zero tensors. Similarly $a_r^{(i)} - \theta_r b^{(i)} \neq 0$ means that the three vectors obtained by putting $i = 1, 2, 3$ are *not all three* zero vectors.]

EXAMPLES.

1. Prove that $A_{rst} = A_{123} e_{rst}$.
2. If we put $A = A_{123}/\sqrt{g}$, show that $A_{rst} = A e_{rst}$, and deduce that A is an invariant.
3. Prove that

$$A_{rst} = a_r^{(i)} A_{st}^{(i)} = \sum_i^3 a_r^{(i)} A_{st}^{(i)}.$$

4. Prove that

$$A_{rst} = a_r^{(1)} A_{st}^{(1)} + a_s^{(1)} A_{tr}^{(1)} + a_t^{(1)} A_{rs}^{(1)} = \frac{1}{2} \delta_{rst}^{mn} p a_m^{(1)} A_n^{(1)}.$$

5. If $A_{rst} = 0$, show that we can choose numbers $\lambda_{(1)}$, $\lambda_{(2)}$, $\lambda_{(3)}$ such that

$$\lambda_{(i)} a_r^{(i)} = 0.$$

If, in addition, $b^{(i)} A_{rs}^{(i)} = 0$, then the quantities $\lambda_{(i)}$ also satisfy the equation

$$\lambda_{(i)} b^{(i)} = 0.$$

5. Plane coordinates.

We have seen that the equation of a plane is of the form

$$a_r x^r = b.$$

Let us suppose that the plane does not pass through the origin, so that b is not zero, and let us divide across by b . The equation of the plane becomes

$$\frac{a_r x^r}{b} = 1.$$

Putting $u_r = a_r/b$, we have finally

$$u_r x^r = 1. \quad (22)$$

Now if we are given the three quantities (u_1, u_2, u_3) then the equation (22) determines the plane uniquely, just as (x^1, x^2, x^3) determine a point uniquely. Consequently we call them, by analogy with x^r , the *rectilinear coordinates of the plane*, and we shall refer to u_r as the *plane variables* in contradistinction to the x^r , which we call the *point variables*. The equation (22) shows us that the u_r form a covariant vector. Thus if we transform from one set of rectilinear coordinates to another, the point variables transform in accordance with the equations

$$\bar{x}^r = c_s^r x^s, \quad (23)$$

whilst the law of transformation of the plane variables is given by

$$\bar{u}_r = \gamma_s^r u_s, \quad (24)$$

γ_s^r being the cofactor of c_s^r in $|c_s^r|$, divided by $|c_s^r|$. A transformation of rectilinear coordinate systems is therefore given by a linear transformation either of point variables or of plane variables.

Let us put

$$u_r = \theta \lambda_r, \quad (25)$$

where λ_r is a given unit vector and θ is a variable parameter. We wish to examine how the plane, whose coordinates are (25), varies as θ varies. The equation of the plane in point variables, or, as it is often called, the *point equation* of the plane is, from (22),

$$\lambda_r x^r = \frac{1}{\theta}.$$

Hence as θ varies we have a *series of parallel planes*, the normal vector to the planes being λ^r . Moreover, $1/\theta$ is the length of the perpendicular from the origin on the plane, and therefore the plane goes off to infinity as θ tends to zero and approaches the origin as θ tends to infinity. We conclude that the plane whose coordinates are $(0, 0, 0)$ is *the plane at infinity*, and if we wish to include planes through the origin we must let u_1, u_2, u_3 tend to infinity in given ratios. We see from (25) that θ must be equal to u , the magnitude of u^r , and hence u is equal to the reciprocal of the length of the perpendicular from the origin on the plane.

The system of planes $(0, 0, t)$, as t varies, has the point equation

$$x^3 = \frac{1}{t},$$

that is, it is the system of planes parallel to the coordinate plane $x^3 = 0$, and the member of the system, given by $t = 1$, passes through the unit point E_3 . We shall call this plane a *unit plane*. We have similar results for the planes parallel to the other two coordinate planes, so that we have *three unit planes* with coordinates $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. If we write the coordinates of the unit planes in the form $e_r^{(i)}$, then

$$e_r^{(i)} = \delta_r^i \dots \dots \dots (26)$$

in each coordinate system. It is easy to see (cf. p. 39) that we can always choose a coordinate system such that any three planes, which have a unique finite point of intersection and which do not pass through the origin, are the unit planes of that system. Let us draw through the origin planes parallel to the three given planes, thus forming a parallelepiped, and let E_1, E_2, E_3 be the corners of the parallelepiped on the edges through O . If we take a coordinate system in which E_1, E_2, E_3 are the unit points, the three given planes are planes through the unit points drawn parallel to the coordinate planes and are therefore the unit planes of the coordinate system.

EXAMPLES.

1. Show that the point of intersection of the unit planes has for point coordinates $(1, 1, 1)$.

2. Show that the length of the perpendicular from O on the plane u_r is $\frac{1}{\sqrt{g^{mn} u_m u_n}}$, and deduce that the perpendiculars on the unit planes are

$$1/\sqrt{g^{11}}, 1/\sqrt{g^{22}}, 1/\sqrt{g^{33}}.$$

3. Show that the point u^r is the pole of the plane u_r with respect to the unit sphere whose centre is O .

4. Prove that the angle φ between the planes u_r and v_r is given by

$$\cos \varphi = \frac{g_{mn} u_m v_n}{u v}.$$

5. Deduce from Ex. 4 that the angles φ_{12} , φ_{23} , φ_{31} between the coordinate planes are given by

$$\cos \varphi_{12} = \frac{g^{12}}{\sqrt{g^{11} g^{22}}}, \quad \cos \varphi_{23} = \frac{g^{23}}{\sqrt{g^{22} g^{33}}}, \quad \cos \varphi_{31} = \frac{g^{31}}{\sqrt{g^{33} g^{11}}}$$

6. Show that to each unit covariant vector there corresponds a plane touching the unit sphere with centre O .

7. If $e_{(i)}$ are the unit points and $e^{(i)}$ the unit planes, show that $e^{(i)}$ is the cofactor of $e_{(i)}$ in the determinant $|e_{(i)}|$.

6. Systems of planes.

If we are given two planes whose coordinates are u_r and v_r , let us examine the plane of coordinates

$$\frac{\lambda u_r + \mu v_r}{\lambda + \mu} \quad (27)$$

The point equation of this plane is

$$\left(\frac{\lambda u_r + \mu v_r}{\lambda + \mu} \right) x^r = 1,$$

or, as it may be written,

$$\lambda (u_r x^r - 1) + \mu (v_r x^r - 1) = 0.$$

Reference to § 3 (p. 57) shows us that it is a plane passing through the intersection of u_r and v_r . Moreover, if it makes angles α , β with the planes u_r , v_r respectively, then

$$\lambda u \sin \alpha + \mu v \sin \beta = 0,$$

or

$$\frac{v \sin \beta}{u \sin \alpha} \quad (28)$$

In particular, the plane $(u_r + k v_r)/(1 + k)$ divides the angle between u_r , v_r into parts α , β such that $k = -u \sin \alpha / v \sin \beta$.

We see that the two planes

$$\frac{u_r + k v_r}{1 + k}, \quad \frac{u_r + k' v_r}{1 + k'} \quad (29)$$

are pairs of planes in a homographic system if k, k' are connected by a linear equation

$$akk' + bk + ck' + d = 0, \quad .$$

and the system is in involution if $b = c$.

If we are given three planes, we may denote them by

$$u_r^{(1)} = u_r^{(1)}, u_r^{(2)}, u_r^{(3)} (30)$$

We see from § 4 (p. 58) that the necessary and sufficient condition that these planes should have a unique finite point of intersection is that

$$|u_r^{(i)}| \neq 0; \quad (31)$$

also any other plane passing through this point may be represented by

$$u_r = \frac{\lambda_{(1)}u_r^{(1)} + \lambda_{(2)}u_r^{(2)} + \lambda_{(3)}u_r^{(3)}}{\lambda_{(1)} + \lambda_{(2)} + \lambda_{(3)}} = \frac{\lambda_{(i)}u_r^{(i)}}{\sum \lambda_{(i)}} . . . (32)$$

EXAMPLES.

1. Prove that the cross ratio of the pencil formed by the planes (29) and u_r, v_r is k/k' .

2. Prove that the planes $u_r, v_r, \frac{u_r + kv_r}{1+k}, \frac{u_r - kv_r}{1-k}$ form a harmonic pencil.

3. Show that the planes $\frac{vu_r \pm uv_r}{v \pm u}$ are the bisectors of the angle between u_r, v_r .

4. Defining parallel planes as planes having a common line of intersection at infinity, show that the system of planes parallel to u_r is $0u_r$, as θ varies.

7. The equation of a point.

We have seen that a linear equation in the point variables is the equation of a plane. Let us examine what is represented by a linear equation in the plane variables. Such a linear equation is of the form

$$a'u_r = b.$$

By (22), p. 61, this equation expresses the condition that the plane u_r should pass through the point a'/b . Hence the linear equation in u_r represents a system of planes passing through a

fixed point, and we call it the plane equation of the point. If b tends to zero, we see that the point a^r/b tends to infinity along the line joining O to a^r , whilst the equation (33) becomes of the form

$$a^r u_r = 0. \quad (34)$$

Thus the homogeneous linear equation (34) in plane variables represents the point at infinity in the direction a^r .

If we are given two linear equations

$$\begin{aligned} a^r u_r &= b, \\ a'^r u_r &= b', \end{aligned} \quad (35)$$

these will be satisfied simultaneously by those planes which pass through both the points a^r/b and a'^r/b' , and (35) will then represent the system of planes through the line joining these points. Thus two linear plane equations represent the straight line joining the two points. If we put

$$A^{rs} = \frac{a^r a'^s}{a^s a'^r}, \quad (36)$$

we have the following cases:

(A) If $A^{rs} \neq 0$, then $a'^r \neq 0$ and the two points are not on a straight line through O . The line joining the points does not pass through the origin.

(B) If $A^{rs} = 0$, the points lie on a line through O and we have two sub-cases:

(i) $\frac{a'^r}{b'} \neq \frac{a^r}{b}$, the points are not coincident and the line passes through the origin.

(ii) $\frac{a'^r}{b'} = \frac{a^r}{b}$, the two points are coincident.

The student will remark the analogy between these results and those of § 3 (p. 56).

If we are given three linear equations

$$a'_{(i)} u_r = b_{(i)}, \quad (37)$$

these will represent three given points and we wish to find the relations between these points. Let us use the notation

$$A^{rst} = \begin{vmatrix} a'_{(1)} & a''_{(1)} & a'_{(1)} \\ a'_{(2)} & a''_{(2)} & a'_{(2)} \\ a'_{(3)} & a''_{(3)} & a'_{(3)} \end{vmatrix}, \quad (38)$$

$$A''_{(1)} = \begin{vmatrix} a'_{(2)} & a''_{(2)} \\ a'_{(3)} & a''_{(3)} \end{vmatrix}, \quad A''_{(2)} = \begin{vmatrix} a'_{(1)} & a''_{(1)} \\ a'_{(3)} & a''_{(3)} \end{vmatrix}, \quad A''_{(3)} = \begin{vmatrix} a'_{(1)} & a''_{(1)} \\ a'_{(2)} & a''_{(2)} \end{vmatrix}$$

We leave the student to prove, by reasoning similar to that of § 4 (p. 58), the following results.

(A) $A^{rst} \neq 0$, the three points lie on a unique plane not passing through the origin.

(B) $A^{rst} = 0$, $A_{(i)}^{rs} \neq 0$.

(i) $b_{(i)} A_{(i)}^{rs} \neq 0$, the three points lie on a unique plane through the origin.

(ii) $b_{(i)} A_{(i)}^{rs} = 0$, the three points lie on a line not passing through the origin.

(C) $A^{rs} = 0$, $A_{(i)}^{rs} = 0$.

(i) $a_{(i)}^r - \theta^r b_{(i)} \neq 0$, the points lie on a line through the origin.

(ii) $a_{(i)}^r - \theta^r b_{(i)} = 0$, the points are all coincident.

The student will observe the complete duality existing between the two sets of point and plane variables. The point in one corresponds to the plane in the other, the line joining two points corresponds to the line of intersection of two planes, the origin corresponds to the plane at infinity and so on. This principle of correspondence between the two sets of variables is known as the *principle of duality*.

EXAMPLES.

1. Show that the equations in plane coordinates analogous to (1) and (3), p. 53, give two forms of the plane equation of a point.

2. Prove that

$$\lambda (a^r u_r - b) + \mu (a'^r u_r - b') = 0$$

is the equation of any point lying on the line joining the points (35).

3. Show that the point in Ex. 2 divides the points (35) in the ratio $\mu b'/\lambda b$.

4. Prove that the point

$$\lambda^{(i)} (a_{(i)}^r u_r - b_{(i)}) = 0$$

is the mean centre of the three points (37) for multiples $\lambda^{(1)} b_{(1)}$, $\lambda^{(2)} b_{(2)}$, $\lambda^{(3)} b_{(3)}$.

EXAMPLES IV.

1. If the equation of the plane is given in the form (3), p. 53, show that the vector $\varepsilon^{r m n} A_m B_n$ is normal to the plane.

2. If the equation of the plane is (1), p. 53, show that the vector $g^{rs} \varepsilon_{smn} (x_{(1)}^m x_{(2)}^n + x_{(2)}^m x_{(3)}^n + x_{(3)}^m x_{(1)}^n)$ is normal to the plane.

3. The coordinates of the foot of the perpendicular from the point x_0^r on the plane $a_r x^r = b$ are

$$x_0^r + (b - a_s x_0^s) a^r / a^2.$$

4. The plane passing through the intersection of the planes $a_r x^r = b = 0$ and $a'_r x^r = b' = 0$ and also through the origin is

$$(b' a_r - b a'_r) x^r = 0.$$

5. If a plane makes angles α , β with two other planes, which are inclined to each other at an angle θ , the lines of intersection of the planes being parallel, prove that

$$\sin^2 \theta = \sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta \cos \theta.$$

6. If $A \neq 0$, prove that the unique point of intersection of the three planes in § 4 (p. 58) is

$$\frac{\varepsilon^{rst} b^{(i)} A_{rst}^{(i)}}{2A}.$$

7. Using the notation of § 4 (p. 58), let us denote by M the matrix

$$\begin{pmatrix} a_{31}^{(1)} & a_{32}^{(1)} & b^{(1)} \\ a_{31}^{(2)} & a_{32}^{(2)} & b^{(2)} \\ a_{31}^{(3)} & a_{32}^{(3)} & b^{(3)} \end{pmatrix}$$

(i) Prove that the conditions $A_{rst} = 0$, $b^{(i)} A_{rs}^{(i)} = 0$ express that all the determinants of the third order in M are zero.

(ii) Prove that the conditions $A_{rst}^{(i)} = 0$, $a_r^{(1)}/b^{(1)} = a_r^{(2)}/b^{(2)} = a_r^{(3)}/b^{(3)}$ express that all the determinants of the second order in M are zero.

8. If $w_r^{(i)}$ are three planes whose coordinates satisfy $|w_r^{(i)}| \neq 0$, show that the transformation to new variables u_r

$$u_r = w_r^{(i)} \tilde{w}_i$$

is such that the given planes are the unit planes of the new system. Hence show that any three covariant vectors, $w_r^{(i)}$, satisfying the condition $|w_r^{(i)}| \neq 0$, can assume the values (1, 0, 0), (0, 1, 0) and (0, 0, 1) if the coordinate system is chosen suitably.

9. Prove that an orthogonal Cartesian system can always be chosen such that any covariant vector shall have two of its components zero.

10. Prove that the vector $\varepsilon^{rst} u_m^{(1)} u_n^{(2)}$ is parallel to the line of intersection of the planes $u_m^{(1)}$, $u_n^{(2)}$.

11. Show that the point, whose equation is

$$\varepsilon^{rst} u_m^{(1)} u_n^{(2)} u_p = 0,$$

is the point at infinity on the line of intersection of $u_p^{(1)}$ and $u_p^{(2)}$.

12. Let $u_r^{(1)}$, $u_r^{(2)}$, $u_r^{(3)}$, $u_r^{(4)}$ be four planes and let us put

$$A_r = \begin{pmatrix} u_r^{(1)} & u_r^{(1)} & u_r^{(1)} \\ u_r^{(2)} & u_r^{(2)} & u_r^{(2)} \\ u_r^{(3)} & u_r^{(3)} & u_r^{(3)} \\ u_r^{(4)} & u_r^{(4)} & u_r^{(4)} \end{pmatrix}$$

Prove that $A_{rst} = \varepsilon^{123} \varepsilon_{rst}$, and deduce that $A = \frac{A_{123}}{|g|}$ is an invariant.

13. Deduce from (32), p. 64, that $A = 0$ is the condition that the four planes of Ex. 12 should meet in a point.

14. If q_{23} is the angle between the coordinate planes $x^2 = 0$ and $x^3 = 0$, prove that

$$\sin^2 q_{23} = \frac{g_{11}}{g g^{22} g^{33}}$$

15. We are given three planes $u_r^{(i)} x^r = 0$ passing through the origin and α , β , γ are the angles between their lines of intersection. By evaluating the invariant

$$\frac{\varepsilon^{rst} u_r^{(1)} u_s^{(2)} u_t^{(3)}}{\left\{ \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \right\}}$$

in the Cartesian system in which these planes are the coordinate planes, prove that

$$\frac{1}{\sqrt{g}} \begin{vmatrix} u_1^{(1)} & u_2^{(1)} & u_3^{(1)} \\ u_1^{(2)} & u_2^{(2)} & u_3^{(2)} \\ u_1^{(3)} & u_2^{(3)} & u_3^{(3)} \end{vmatrix} = \left\{ \prod_s g^{mn} u_m^{(s)} u_n^{(s)} \right\}^{1/2} \left\{ \frac{1 - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma + 2 \cos \alpha \cos \beta \cos \gamma}{\sin \alpha \sin \beta \sin \gamma} \right\}.$$

16. The plane coordinates of four planes are $u_r^{(1)}$, $u_r^{(2)}$, $u_r^{(3)}$, $u_r^{(4)}$ and we put

$$\begin{aligned} A^{(1)} &= \varepsilon^{rst} u_r^{(4)} u_s^{(2)} u_t^{(3)}, & A^{(2)} &= \varepsilon^{rst} u_r^{(1)} u_s^{(4)} u_t^{(3)}, \\ A^{(3)} &= \varepsilon^{rst} u_r^{(1)} u_s^{(2)} u_t^{(4)}, & A^{(4)} &= -\varepsilon^{rst} u_r^{(1)} u_s^{(2)} u_t^{(3)}. \end{aligned}$$

Show that

$$\frac{1}{\sqrt{g}} \begin{vmatrix} u_1^{(1)} & u_2^{(1)} & u_3^{(1)} & 1 \\ u_1^{(2)} & u_2^{(2)} & u_3^{(2)} & 1 \\ u_1^{(3)} & u_2^{(3)} & u_3^{(3)} & 1 \\ u_1^{(4)} & u_2^{(4)} & u_3^{(4)} & 1 \end{vmatrix} = \sum_{\alpha} A^{(\alpha)}, \quad \frac{1}{\sqrt{g}} \begin{vmatrix} u_2^{(1)} & u_3^{(1)} \\ u_2^{(2)} & u_3^{(2)} \\ u_2^{(3)} & u_3^{(3)} \\ u_2^{(4)} & u_3^{(4)} \end{vmatrix} = A^{(1)}, \text{ etc. } \dots$$

17. By calculating the invariant

$$\sum A^{(\alpha)} \quad \bar{\Pi} A^{(\alpha)}$$

in the system in which $u_1^{(1)}$, $u_2^{(2)}$, $u_3^{(3)}$ are the unit planes, show that this invariant equals $6V$, where V is the volume of the tetrahedron formed by the four planes.

CHAPTER V

THE STRAIGHT LINE

1. The point equations of the straight line.

There are three different forms in which the equations of a straight line can be written.

First, let us be given two points $x_{(1)}^r$ and $x_{(2)}^r$ on the line. The point

$$x^r = \frac{\lambda x_{(1)}^r + \mu x_{(2)}^r}{\lambda + \mu} \quad \dots \quad (1)$$

is the mean centre of these two points for multiples λ and μ . In other words, it lies on the straight line and divides the given points in the ratio μ/λ . For different values of λ and μ (1) represents every point on the straight line.

Secondly, let x_0'' be a given point on the straight line and let λ'' be the unit vector defining the direction of the line. Then any point of the line is given by

$$x'' = x_0'' + \varrho \lambda'', \quad (2)$$

where ϱ is the distance from x_0'' to x'' along the line.

Lastly, the straight line may be given as the intersection of two planes, whose equations are

$$\begin{aligned} a^{(1)} x'' &= b^{(1)}, \\ a^{(2)} x'' &= b^{(2)}. \end{aligned} \quad \cdot \cdot \cdot \cdot \cdot \cdot \quad (3)$$

These give the third form into which the equations may be put.

The particular form to be used will depend on the particular problem. If we are dealing with systems of points on a single line then (1) will be found a convenient form; if we are dealing with several lines the form (2) will usually be most suitable; whilst (3) will be of use only in dealing with planes passing through the line.

EXAMPLES.

1. If d is the distance between $x_{(1)}''$ and $x_{(2)}''$ show that $\lambda'' = \frac{x_{(2)}'' - x_{(1)}''}{d}$, and show that (1) takes the form (2) by putting $\mu/(\lambda + \mu) = \varrho/d$.
2. By writing (2) in the form $x'' = (1 - \varrho) x_0'' + \varrho (x_0'' + \lambda'')$, show that (2) is already implicitly in the form (1).
3. Show that in (3) λ'' is proportional to $a^{r'm} a_m^{(1)} a_n^{(2)}$.
4. Find the ratios of $\lambda : \mu$ for which the point (1) lies (i) on the plane $a_r x'' = b$, (ii) on the coordinate planes.

2. The relations of two straight lines.

We shall here employ the form (2) for the equations of the straight lines. Let them be

$$\begin{aligned} x'' &= x_0'' + \varrho \lambda'', \\ x'' &= x_0'' + \varrho' \lambda''', \end{aligned} \quad \cdot \cdot \cdot \cdot \cdot \cdot \quad (4)$$

Since two parallel vectors have the same components, the straight lines through the origin parallel to (4) are

$$x'' = \varrho \lambda'', \quad x'' = \varrho' \lambda''',$$

and the angle between the lines is equal to the angle between their parallels through O . Hence the angle θ between the lines is given by

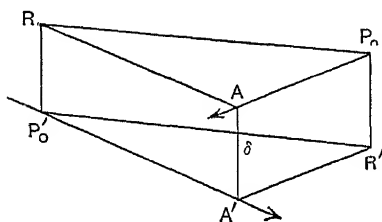
$$\cos \theta = g_{mn} \lambda^m \lambda'^n = \lambda_m \lambda'^m \dots \dots \dots (5)$$

The two straight lines are perpendicular if

$$g_{mn} \lambda^m \lambda'^n = 0,$$

and are parallel if $\lambda'^r = \lambda^r$.

Let us now consider the shortest distance between the two straight lines. Let $P_0 A$ and $P'_0 A'$ be the lines and through each of them let us draw planes $P_0 A R$ and $P'_0 A' R'$ parallel to the other. These planes are obviously parallel to each other. We next draw planes $P_0 R' A A'$ and $P'_0 R A A'$ perpendicular to these planes and intersecting the lines in A and A' . It is clear that $A A'$ is not only perpendicular to both lines



but is also perpendicular to both planes. Also it is easily seen that $A A'$ is the shortest distance between the lines, for the distance between any two other points of the lines is greater than the perpendicular distance between the two parallel planes.

Let the coordinates of P_0 , P'_0 be x_0^r , $x_0'^r$ respectively and δ the shortest distance. The vector

$$\frac{\lambda_m \lambda'_m}{\sin \theta} \quad (6)$$

is the unit vector orthogonal to λ^r and λ'^r , that is, to the two straight lines. But δ is the projection of $P_0 P'_0$ on this direction. The shortest distance between the lines is therefore

$$\delta = \frac{(x_0^r - x_0'^r) \lambda^m \lambda'_m}{\sin \theta} \dots \dots \dots (7)$$

EXAMPLES.

1. Show that $\sin^2 \theta = \delta_{rs}^m \lambda^r \lambda'^s \lambda_m \lambda'_m$.
2. The coordinates of A and A' are

$$x_0^r + \frac{\lambda^r}{\sin^2 \theta} (\lambda'_m \cos \theta - \lambda_m) (x_0^m - x_0'^m),$$

$$x_0'^r + \frac{\lambda'^r}{\sin^2 \theta} (\lambda_m \cos \theta - \lambda'_m) (x_0^m - x_0'^m).$$

3. Prove that the equations of the two parallel planes are

$$\varepsilon_{mnp} \lambda^m \lambda'^n (x^p - x_0^p) = 0, \quad \varepsilon_{mnp} \lambda^m \lambda'^n (x^p - x_0'^p) = 0.$$

4. Prove that the equations of the perpendicular planes are

$$(\lambda_r \cos \theta - \lambda'_r)(x'' - x'_0) = 0, \quad (\lambda'_r \cos \theta - \lambda_r)(x'' - x'_0) = 0.$$

5. Prove that the two lines intersect if

$$\varepsilon_{rmn} x_0^r \lambda^m \lambda'^n = \varepsilon_{rmn} x_0^{r'} \lambda'^m \lambda'^n.$$

3. The six coordinates of a straight line.

Let P_0L be a straight line whose equation is given in the form

$$x'' = x_0^r + \varrho \lambda^r \quad . \quad . \quad . \quad (8)$$

Let us form the quantities

$$\mu_r = \varepsilon_{rmn} \lambda^m x_0^n. \quad (9)$$

These form a covariant vector and it is easily verified that they are independent of the particular point x_0^r that we take on the line. The two vectors λ^r and μ_r determine the straight line uniquely, for we shall now find the equations of the line in terms of them. If we multiply (9) by $\varepsilon^{srp} \lambda_p$ and sum with respect to r from 1 to 3, we get

$$\begin{aligned} \varepsilon^{srp} \mu_r \lambda_p &= \varepsilon^{srp} \varepsilon_{rmn} \lambda^m x_0^n \lambda_p = -\delta_{mn}^s \lambda^m x_0^n \lambda_p \\ &= -(\lambda^s x_0^p - \lambda^p x_0^s) \lambda_p = x_0^s - (\lambda_p x_0^p) \lambda^s, \end{aligned}$$

which shows that the vector $\varepsilon^{rst} \mu_s \lambda_t$ is a point on the straight line. The equations of the straight line can therefore be written

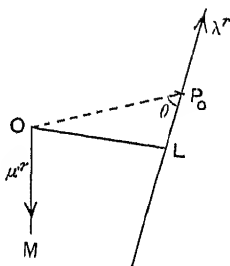
$$x^r = \varepsilon^{rst} \mu_s \lambda_t + \varrho \lambda^r. \quad (10)$$

The two vectors λ^r , μ_r are called *the six coordinates of the straight line*. They satisfy the relations

$$\begin{aligned} \lambda_m \lambda^m &= 1, \\ \lambda^m \mu_m &= 0, \end{aligned} \quad (11)$$

which state that λ^r is a unit vector and that μ^r is orthogonal to λ^r .

Interpreted geometrically, λ^r is the versor giving the direction of the line. The vector μ^r is perpendicular to both λ^r and x_0^r and its direction is such that the triad $(\lambda^r, x_0^r, \mu^r)$ is positive. Also its magnitude is $OP_0 \sin \theta$, that is, it equals OL where L



is the foot of the perpendicular from O on the line. Hence if we draw OM perpendicular to the plane OLP_0 , in such a direction that LP_0 , OP_0 , OM form a positive triad, and cut off OM equal to OL , then the point M has μ^r for its coordinates.

EXAMPLES.

1. Prove that the plane $\mu_r x^r = 0$ is the plane through O containing the straight line.
2. Show that the coordinates of L are $\varepsilon^{rmn} \mu_m \lambda_n$.
3. If the straight line passes through the origin, show that $\mu_r = 0$.
4. Show that $g^{rs} \varepsilon_{smn} \mu^m x_0^n = x_0^r g_{mn} x_0^m \lambda^n - \lambda^r g_{mn} x_0^m x_0^n$.
5. Find the six coordinates of each of the coordinate axes.

4. The plane equations of a straight line.

We may also express the equations of a straight line in terms of plane coordinates. There are two forms in which the equations may be written and these are analogous to the forms (1) and (3), pp. 68, 69.

If $u_r^{(1)}$ and $u_r^{(2)}$ are the coordinates of any two planes passing through the straight line, then the plane

$$u_r = \frac{\lambda u_r^{(1)} + \mu u_r^{(2)}}{\lambda + \mu} \dots \dots \dots (12)$$

is any plane passing through the given line. This form of the equations is of value in dealing with systems of planes through the straight line.

The straight line may be given as the join of two points whose plane equations are

$$\begin{aligned} a_{(1)}^r u_r &= b_{(1)}, \\ a_{(2)}^r u_r &= b_{(2)}. \end{aligned} \dots \dots \dots (13)$$

This is the second form of the equations, and can be used when we wish to deal with systems of points of the line in plane coordinates.

EXAMPLES.

1. Show that as $\mu/\lambda \rightarrow -1$ the plane (12) tends to the plane through O containing the straight line.
2. Deduce from Ex. 1 that the point equation of the plane through O and the line is $(u_r^{(1)} - u_r^{(2)}) x^r = 0$.
3. Show that the vector $\varepsilon^{rmn} u_m^{(1)} u_n^{(2)}$ is parallel to the straight line and that its magnitude is $u^{(1)} u^{(2)} \sin \theta$, θ being the angle between $u_r^{(1)}$ and $u_r^{(2)}$.

EXAMPLES V

1. If a, b are the opposite sides of a tetrahedron, d the shortest distance between them and θ the angle between them, prove that the volume of the tetrahedron is

$$V = \frac{1}{6} a b d \sin \theta.$$

[Let $x_0', x_0'', x_0' + a \lambda^r, x_0'' + b \lambda'^r$ be the vertices. Then

$$6V = \varepsilon_{rst} (x_0' - x_0'') (x_0^s - x_0'^s + a \lambda^s) b \lambda'^t = \varepsilon_{rst} (x_0' - x_0'') \lambda^s \lambda'^t a b.]$$

2. Show that the perpendicular distance between two parallel lines $x^r = x_0^r + \rho \lambda^r$ and $x^r = x_0'^r + \rho' \lambda'^r$ is δ where

$$\delta^2 = (g_{mn} - \lambda_m \lambda_n) (x_0^m - x_0'^m) (x_0^m - x_0'^m).$$

3. If a straight line is defined by two points $x_{(1)}^r, x_{(2)}^r$ show that its six coordinates are given by

$$\lambda^r = \frac{x_{(2)}^r - x_{(1)}^r}{d}, \quad \mu_r = \frac{\varepsilon_{rmn} x_{(2)}^m x_{(1)}^n}{d},$$

where d is the distance between $x_{(1)}^r$ and $x_{(2)}^r$.

4. If a straight line is determined by two planes $u_r^{(1)}, u_r^{(2)}$, show that its six coordinates are given by

$$\lambda^r = \frac{\varepsilon^{rmn} u_m^{(1)} u_n^{(2)}}{k}, \quad \mu_r = - \frac{u_r^{(1)} - u_r^{(2)}}{k},$$

where $k = u^{(1)} u^{(2)} \sin \theta$, $u^{(1)}, u^{(2)}$ being the magnitudes of $u_r^{(1)}$ and $u_r^{(2)}$ and θ being the angle between them.

[Use the results of Exs. 2. and 3 of § 4, pp. 72, 73.]

5. If the six coordinates of two lines are λ^r, μ_r and λ'^r, μ'_r , show that the shortest distance between the lines is

$$\delta = \frac{\lambda^m \mu'_m + \lambda'^m \mu_m}{\sin \theta},$$

θ being the angle between the lines.

6. The condition that two lines intersect is $\lambda^m \mu'_m + \lambda'^m \mu_m = 0$.

7. Show that the planes

$$\varepsilon_{rmn} \lambda^m x^n = \mu_r, \quad \mu_r x^r = 0$$

all pass through the line (λ^r, μ_r) .

8. Show that any plane through the straight line (λ^r, μ_r) is given by

$$u_r = \frac{1}{\mu^2} \varepsilon_{rst} \mu^s \lambda^t + k \mu_r$$

as k varies. Show that $k = \cot \alpha / \mu^2$, where α is the angle between u_r and μ_r . For $k = 0$ we have the plane passing through the line and perpendicular to OL .

9. The six coordinates of a line may also be looked upon as a *contra-variant vector* λ^r and the non-zero components of a *skew-symmetric double tensor* $\mu^{rs} = \lambda^r x_0^s - \lambda^s x_0^r$, with the relations $\lambda_m \lambda^m = 1$, $\varepsilon_{rst} \lambda^r \mu^{st} = 0$. Show that the equations of the line are $x^r = \mu^{sr} \lambda_s + \varrho \lambda^r$.

10. With the notation of Ex. 9, show that the shortest distance between two lines is

$$\delta = \varepsilon_{rst} (\lambda^r \mu^{st} + \lambda'^r \mu'^{st}) / 2 \sin \theta.$$

11. If we have two straight lines $x^r = x_0^r + \varrho \lambda^r$, $x^r = x_0^r + \varrho' \mu'^r$, both passing through the point x_0^r , show that the equations of the bisectors of the angle between them are

$$x^r = x_0^r + t (\lambda^r \pm \mu'^r),$$

and show that distances along these lines are measured by $2t \cos \theta/2$ and $2t \sin \theta/2$, θ being the angle between λ^r and μ'^r .

12. If OA, OB, OC are three directions from O given by the unit vectors λ^r, μ^r, ν^r and OA', OB', OC' are the bisectors of the angles between these directions, show that the planes AOA', BOB', COC' all pass through the line $x^r = t (\lambda^r + \mu^r + \nu^r)$.

13. Show that the length of the perpendicular from x^r on the line $x^r = x_0^r + \varrho \lambda^r$ is δ where

$$\delta^2 = (g_{mn} g_{rs} - g_{mr} g_{ns}) (x^m - x_0^m) (x^n - x_0^n) \lambda^r \lambda^s.$$

14. Show that the equation of the plane through the line $x^r = x_0^r + \varrho \lambda^r$ and parallel to the line $x^r = \varrho' \mu'^r$ is $\varepsilon_{rmn} (x^r - x_0^r) \lambda^m \mu'^n = 0$.

15. If we are given two straight lines as the intersection of the pairs of planes $u_r^{(1)}, u_r^{(2)}$ and $u_r^{(3)}, u_r^{(4)}$, show that

$$\tan \theta = - \frac{1}{\sqrt{g}} \begin{vmatrix} u_1^{(1)} & u_1^{(2)} & u_1^{(3)} & u_1^{(4)} \\ u_2^{(1)} & u_2^{(2)} & u_2^{(3)} & u_2^{(4)} \\ u_3^{(1)} & u_3^{(2)} & u_3^{(3)} & u_3^{(4)} \\ 1 & 1 & 1 & 1 \end{vmatrix} [\varepsilon_{rst} \varepsilon^{r'm'n} u_m^{(1)} u_n^{(2)} \varepsilon^{s'p'q} u_p^{(3)} u_q^{(4)}],$$

where δ is the shortest distance and θ the angle between the lines.

[Use Exs. 4 and 5.]

16. The condition that the straight lines of Ex. 15 be coplanar is

$$\begin{vmatrix} u_1^{(1)} & u_1^{(2)} & u_1^{(3)} & u_1^{(4)} \\ u_2^{(1)} & u_2^{(2)} & & \\ u_3^{(1)} & & & \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

17. Show that the points

$$\frac{x_1^r + k x_2^r}{1 + k}, \quad \frac{x_1^r + k' x_2^r}{1 + k'}$$

form pairs of points of a homographic range on the line joining x_1^r and x_2^r if k and k' are connected by a relation of the form

$$a k k' + b k + c k' + d = 0.$$

CHAPTER VI

THE QUADRIC CONE AND THE CONIC

1. The equation of a quadric cone.

A quadric cone is the locus traced out by a family of lines which all pass through a point x_0^r and whose versors λ^r satisfy a quadratic relation of the form

$$a_{mn} \lambda^m \lambda^n = 0, \quad (1)$$

where a_{mn} are the components of a symmetric double system. The point x_0^r is called *the vertex* and each of the lines is called *a generator of the cone*.

If x^r is a point on the cone, the line joining it to the vertex is obviously a generator; but the equations of this line are

$$x^r = x_0^r + \rho \lambda^r.$$

Hence we deduce that x^r satisfies the equation

$$(x^m - x_0^m)(x^n - x_0^n) = 0. \quad (2)$$

This is therefore *the equation of the cone*. We see, as we would expect, that if a point satisfies the equation (2) then every point on the line joining it to the vertex also satisfies (2). Writing out (2) in full, we have

$$a_{mn} x^m x^n - 2 a_{mn} x^m x_0^n + a_{mn} x_0^m x_0^n = 0. \quad (3)$$

As we shall deal in this chapter only with cones having a common vertex, we may suppose that this vertex is the origin. In this case equation (3) takes the simpler form

$$a_{mn} x^m x^n = 0. \quad (4)$$

When we change to a new coordinate system \bar{x}^r the equation of the cone is still of the form

$$\bar{a}_{mn} \bar{x}^m \bar{x}^n = 0.$$

Hence we conclude that

$$\bar{a}_{mn} \bar{x}^m \bar{x}^n = \lambda a_{mn} x^m x^n,$$

where λ is some factor. We can always suppose that the system \bar{a}_{mn} is multiplied by such a factor that $\lambda = 1$, that is, $a_{mn} x^m x^n$ is an absolute invariant under a transformation of coordinates.

Consequently, since a_{mn} is symmetric, we conclude that a_{mn} is a double covariant tensor except for a possible factor λ . In other words, every quadric cone determines a double covariant tensor.

EXAMPLES.

1. Show that

$$a_{mn} x^m x^n + 2b_n x^n + c \equiv a_{mn} (x^m + \alpha^{ms} b_s) (x^n + \alpha^{nt} b_t) + \frac{\Delta}{A},$$

where α^{mn} is the cofactor of a_{mn} in $|a_{mn}|$, divided by $|a_{mn}|$, and

$$A = |a_{mn}|, \quad \Delta = \begin{vmatrix} a_{mn} & b_m \\ b_n & c \end{vmatrix}$$

Let us denote by A^{mn} the cofactor of a_{mn} in A , so that $\alpha^{mn} = A_{mn}/A$. We have

$$a_{rm} \alpha^{ms} = \delta_r^s, \quad \Delta = cA - A^{mn} b_m b_n.$$

Hence

$$\begin{aligned} & a_{mn} (x^m + \alpha^{ms} b_s) (x^n + \alpha^{nt} b_t) \\ &= a_{mn} x^m x^n + \delta_n^s b_s x^n + \delta_m^t b_t x^m + \delta_n^s \alpha^{nt} b_s b_t \\ &= a_{mn} x^m x^n + 2b_m x^m + \alpha^{mn} b_m b_n. \end{aligned}$$

Also

$$\frac{\Delta}{A} = c - \frac{A^{mn}}{A} b_m b_n = c - \alpha^{mn} b_m b_n,$$

and our result follows immediately. The quantity Δ is usually called the *discriminant* of $a_{mn} x^m x^n + 2b_m x^m + c$.

2. Prove that $\Delta = 0$ is the condition that $a_{mn} x^m x^n + 2b_m x^m + c = 0$ should be the equation of a cone.

3. If $\Delta = 0$, show that the vertex of the cone is $-\alpha^{rs} b_s$ and that $c = \alpha^{mn} b_m b_n$.

4. If we put $F(x) = a_{mn} x^m x^n + 2b_m x^m + c$, show that the equations $\frac{\partial F}{\partial x^r} = 0$ give the vertex of the cone.

5. Show that any straight line not passing through O cuts the cone in two and only two points.

2. The equation of a conic.

Let us now examine the equation corresponding to (2) in plane variables. It is

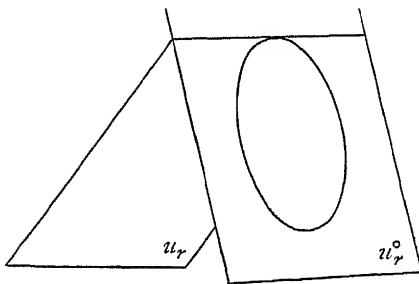
$$a^{mn} (u_m - u_m^0) (u_n - u_n^0) = 0, \quad (5)$$

where u_r^0 is a given plane. We see that a^{mn} is a symmetric *contra-variant* tensor except for an unimportant factor.

The planes whose coordinates satisfy (5) will form a family of planes satisfying some geometrical condition. Let u_r be any plane, other than u_r^0 , which satisfies (5). We see that the plane

$$\frac{u_r + \lambda u_r^0}{1 + \lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

also satisfies (5) no matter what value is given to λ . Now as λ varies the plane (6) always passes through the intersection of u_r and u_r^0 , that is, every plane through this line of intersection satisfies (5). If we take a series of planes u_r , all satisfying (5), their lines of intersection with u_r^0 will give a family of lines in the latter plane and this family of lines will envelope a certain curve in the plane u_r^0 . Moreover this curve is such that all planes through any tangent line to it in the plane u_r^0 satisfy the equation (5). Consequently the equation (5) expresses the condition that the plane u_r touches a given curve in the plane u_r^0 , and we may say that (5) is the *tangential equation* or *plane equation* of the curve.



If we take any two planes $u_r^{(1)}$ and $u_r^{(2)}$, whose line of intersection does not lie in u_r^0 , then $(u_r^{(1)} + k u_r^{(2)})/(1 + k)$ is any plane through their intersection. This plane will touch the curve if its coordinates satisfy (5), that is, if

$$a^{mn} (u_m^{(1)} - u_m^0) (u_n^{(1)} - u_n^0) + 2k a^{mn} (u_m^{(1)} - u_m^0) (u_n^{(2)} - u_n^0) + k^2 a^{mn} (u_m^{(2)} - u_m^0) (u_n^{(2)} - u_n^0) = 0.$$

Hence through any line in space two and only two planes can in general be drawn to touch the curve, and we conclude that it is a curve of the second class or a *conic*.

The equation (5) is therefore the plane equation of a conic in the plane u_r^0 . In other words, every double contravariant tensor determines a conic. We see that the theorems relating to cones with a given point as vertex there correspond dual theorems relating to conics lying in a given plane. We shall work out the theory connected with cones in point coordinates and leave the development of the dual theorems in plane coordinates to the student, the algebraical work in the two cases being exactly the same.

If we consider the equation

$$a^{mn} u_m u_n = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

this represents a conic in the plane whose coordinates are $(0, 0, 0)$. Hence (7) is the equation of a conic in the plane at infinity, and therefore to cones with their vertices at the origin will correspond conics lying in the plane at infinity.

EXAMPLES.

1. If we write

$$A = |a^{mn}|, \quad \Delta = \begin{vmatrix} a^{mn} & b^n \\ b^n & c \end{vmatrix}$$

Show that

$$a^{mn} u_m u_n + 2b^m u_m + c \equiv a^{mn} (u_m + \alpha_{ms} b^s) (u_n + \alpha_{nt} b^t) + \Delta/A,$$

where α_{rs} is the cofactor of a^{rs} in A , divided by A .

2. Show that $\Delta = 0$ is the condition that $a^{mn} u_m u_n + 2b^m u_m + c = 0$ should be the equation of a conic.

3. If $\Delta = 0$, show that the plane of the conic is $-\alpha_{rs} b^s$, and that $c = \alpha_{mn} b^m b^n$.

4. If we put $\Phi(u) = a^{mn} u_m u_n + 2b^m u_m + c$, show that the equations $\frac{\partial \Phi}{\partial u^r} = 0$ give the plane of the conic.

3. The tangent plane to a cone.

Let x'_0 be a point on the cone

$$a_{mn} x^m x^n = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and let x^r be any other point. Then $\frac{x'_0 + kx^r}{1+k}$ is any point on the line joining these points, and this will lie on the cone provided that

$$2k a_{mn} x^m x'_0^n + k^2 a_{mn} x^m x^n = 0. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This is a quadratic equation for k , the roots of which give the points of intersection of the line joining x'_0 and x^r with the cone. One of the roots is zero, as we would expect since x'_0 is on the cone. The line joining x'_0 , x^r will touch the cone if the second root is zero, that is, if

$$a_{mn} x^m x'_0^n = 0. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

This is the equation of a plane and it is such that every line lying in it and passing through x'_0 touches the cone. It is therefore the equation of the *tangent plane to the cone at the point x'_0* . We notice that the tangent plane at x'_0 contains every point of the generator

$$x^r = \theta x'_0.$$

Moreover the tangent planes at every point of this generator are coincident, or, in other words, the plane (10) has a *line of contact* with the cone, namely, the generator.

We can use this result to find the plane equation of the conic in which the cone is cut by the plane at infinity. If x'_0 is any point on the cone, (10) is the equation of the tangent plane at this point and

$$a_{mn} x'^m_0 x'^n_0 = 1 \dots \dots \dots (11)$$

is a parallel plane. Now the line of intersection of the tangent plane with the plane at infinity is a tangent line to the conic section, and therefore the plane (11), which has the same intersection with the plane at infinity, also touches the conic. The plane (11) will be the plane u_r if

$$a_{rn} x'^n_0 = u_r.$$

Solving this equation for x'_0 , we get

$$x'_0 = \alpha'^s u_s,$$

where α'^s is the cofactor of a_{mn} in $|a_{mn}|$; divided by $|a_{mn}|$. But x'_0 is any point on the cone and hence satisfies the equation

$$a_{mn} x'^m_0 x'^n_0 = 0.$$

We therefore deduce that

$$\alpha'^m u_m u_n = 0.$$

This, we know, is the plane equation of a conic in the plane at infinity and must be the section of the cone (8) by the plane at infinity since it is touched by every tangent plane of the cone.

EXAMPLES.

1. Prove that the six coordinates of any generator of the cone are given by $\mu_r = 0$, λ^r , where $a_{mn} \lambda^m \lambda^n = 0$.

2. If we draw tangent planes from the point x'_0 to the cone, prove that the plane containing the lines of contact has for its equation

$$a_{mn} x'^m_0 x'^n_0 = 0.$$

3. Show that the point of contact of the plane u^0_r , which touches the conic $\alpha'^m u_m u_n = 0$, is given by the equation

$$\alpha'^s u_r u^0_s = 0;$$

show that to a generator of the cone in point variables there corresponds a tangent line to the conic in the dual theorem.

4. Deduce from Ex. 3 that the cone formed by lines joining the origin to the conic $\alpha^{mn} u_m u_n = 0$ has for its equation in point variables $a_{mn} x^m x^n = 0$, where a_{mn} is the cofactor of α^{mn} in $|\alpha^{mn}|$, divided by $|\alpha^{mn}|$.

5. State and prove the dual theorem of Ex. 2.

4. Poles and polar planes with respect to a cone.

Let P , coordinates $x_{(1)}^r$, and Q , coordinates $x_{(2)}^r$, be any two points. The point $(x_{(1)}^r + k x_{(2)}^r)/(1+k)$, which divides PQ in the ratio k , lies on the cone if

$$a_{mn} x_{(1)}^m x_{(1)}^n + 2k a_{mn} x_{(1)}^m x_{(2)}^n + k^2 a_{mn} x_{(2)}^m x_{(2)}^n = 0, \quad (12)$$

and the roots of this equation in k are the ratios in which the two points of intersection of PQ with the cone divide the segment PQ .

If the range is harmonic, the roots are equal in magnitude but opposite in sign. Thus

$$a_{mn} x_{(1)}^m x_{(2)}^n = 0,$$

and we see that $x_{(2)}^r$ lies on the plane

$$a_{mn} x^m x_{(1)}^n = 0. \quad \dots \dots \dots (13)$$

This plane is therefore the locus of the harmonic conjugates of P , and we call it the *polar plane of P with respect to the cone*. If two points P and Q are such that the polar plane of one passes through the other, the points are said to be *conjugate points with respect to the cone*. The condition is

$$a_{mn} x_{(1)}^m x_{(2)}^n = 0. \quad \dots \dots \dots (14)$$

We note that the polar planes of all points, which lie on a straight line through O , coincide and we call this plane the *conjugate plane of the given line*. Also if P and Q are conjugate points with respect to the cone then all the points of the line OP are conjugate to every point on OQ . Two such lines are said to be *conjugate diameters*. The condition that two directions $\lambda_{(1)}^r$ and $\lambda_{(2)}^r$ should be conjugate diameters is

$$a_{mn} \lambda_{(1)}^m \lambda_{(2)}^n = 0.$$

EXAMPLES.

1. By expressing the condition that the roots of (12) may be equal, show that the equation of the pair of tangent planes from $x_{(1)}^r$ to the cone is $(a_{mn} x_{(1)}^m x_{(1)}^n)^2 = (a_{mn} x_{(1)}^m x_{(1)}^n) (a_{rs} x^r x^s)$.

2. Find the locus of a point Q such that $PRQS$ has a given cross-ratio λ , where P is the fixed point x_0^r and R, S are the intersections of PQ with the cone.

[The ratio of the roots of (12) equals λ .]

3. Show that the equation of the pole of u_r^0 with respect to the conic $\alpha^{mn} u_m u_n = 0$ is $\alpha^{rs} u_r u_s = 0$.

4. Show that the poles of all parallel planes, with respect to the conic $\alpha^{mn} u_m u_n = 0$, coincide and if p and q are two conjugate planes with respect to the conic then every plane parallel to p is conjugate to every plane parallel to q .

5. The canonical equation of a cone.

Let P be any point which does not lie on the cone $a_{mn} x^m x^n = 0$. If the coordinates of P are x_1^r , the polar plane of P is

$$a_{mn} x_1^m x_1^n = 0,$$

and the point P does not lie on this plane since $a_{mn} x_1^m x_1^n \neq 0$. Hence, if we take any two points x_2^r and x_3^r lying in the polar plane but not collinear with O , then x_1^r, x_2^r, x_3^r are three independent points which may be taken as unit points in a new coordinate system. Now

$$a_{mn} x_1^m x_2^n = a_{mn} x_1^m x_3^n = 0,$$

and these are invariant equations. Therefore, if we transform to the new axes and denote the new quantities by dashes, we have

$$x_1'^r = \delta_1^r, \quad x_2'^r = \delta_2^r, \quad x_3'^r = \delta_3^r,$$

and

$$a'_{12} = a'_{13} = 0.$$

The equation of the cone becomes

$$a'_{11} (x'^1)^2 + a'_{22} (x'^2)^2 + 2a'_{23} x'^2 x'^3 + a'_{33} (x'^3)^2 = 0. \quad (15)$$

If $a'_{11} = a'_{22} = a'_{33} = 0$, the cone reduces to $a'_{11} (x'^1)^2 = 0$, that is, to two coincident planes $x'^1 = 0$, and we proceed no further.

If these conditions are not satisfied, then the section of the cone by the plane $x'^1 = 0$ is the lines

$$a'_{22} (x'^2)^2 + 2 a'_{23} x'^2 x'^3 + a'_{33} (x'^3)^2 = 0. \quad \dots (16')$$

In this plane we can obviously choose a point Q not lying on the lines, and then take a point R on the plane and conjugate to Q with respect to (16), that is, OQ and OR form a harmonic pencil with the lines (16). The three points P , Q , R are independent and may be taken as unit points in a certain system of coordinates \bar{x} . We shall then have in the new system

$$\bar{a}_{12} = \bar{a}_{23} = \bar{a}_{31} = 0,$$

and the equation of the cone becomes

$$\bar{a}_{11} (\bar{x}^1)^2 + \bar{a}_{22} (\bar{x}^2)^2 + \bar{a}_{33} (\bar{x}^3)^2 = 0, \quad \dots (17)$$

where the \bar{a} 's may have zero values. This form of the equation of the cone is called *the canonical form*. Moreover the number of such coordinate systems is infinite as both x'_1 and x'_2 are arbitrary points. We notice that we may take any points on the lines OP , OQ , OR as unit points and the form will still be canonical. In particular, we may choose the points such that the non-zero coefficients of the form (17) may all be plus or minus unity. Thus the equation of the cone may take the special canonical form

$$\varepsilon_1 (\bar{x}^1)^2 + \varepsilon_2 (\bar{x}^2)^2 + \varepsilon_3 (\bar{x}^3)^2 = 0, \quad \dots (18)$$

where ε_1 , for example, is either 0 or is ± 1 and has the same sign as \bar{a}_{11} . If the original form $a_{mn} x^m x^n$ is positive definite then so must the left hand side of (18), that is, all the ε 's must be plus unity.

This result shows that *we can choose in an infinite number of ways a coordinate system in which any symmetric covariant tensor a_{mn} will have*

$$a_{mn} = 0, \quad (m \neq n).$$

EXAMPLES.

1. If we are given a positive definite quadratic form, $g_{mn} x^m x^n$, show that there exists an infinite number of coordinate systems in which it gives the distance of x^r from the origin.

If we take any coordinate system x^r and let

$$\varphi = g_{mn} x^m x^n, \dots \dots \dots (\alpha)$$

we know that we can choose a coordinate system \bar{x}^r such that

$$\bar{\varphi} = (\bar{x}^1)^2 + (\bar{x}^2)^2 + (\bar{x}^3)^2, \quad (\beta)$$

since the form is positive definite. Thus by a linear transformation of variables, which we may call T , we can transform (α) into (β) ; hence the inverse transformation T^{-1} transforms (β) into (α) . Let us now take \bar{x}^r as being any rectangular Cartesian system of coordinates and let us make the transformation T^{-1} to new variables x^r , so that (β) transforms into (α) . But $\bar{\varphi}$ measures the distance from \bar{x}^r to O . Therefore in the new coordinate system φ also measures the distance from O to x^r , and we see that the number of systems available is infinite.

2. Show that we can choose in an infinite number of ways a *Cartesian* system of coordinates in which the equation of the cone assumes the canonical form (17).

3. Show that we can reduce the equation of the cone to the form (17) and at the same time take any plane, which does not touch the cone but passes through the origin, as the plane $\bar{x}^3 = 0$.

4. Show that the conic $\alpha^{mn} u_m u_n = 0$ can be transformed in an infinite number of ways to the canonical form

$$\bar{\alpha}^{11} (\bar{u}_1)^2 + \bar{\alpha}^{22} (\bar{u}_2)^2 + \bar{\alpha}^{33} (\bar{u}_3)^2 = 0.$$

5. Show that the equation of the section of the cone

$$a_{11} (x^1)^2 + a_{22} (x^2)^2 + a_{33} (x^3)^2 = 0$$

by the plane at infinity is

$$\frac{1}{a_{11}} (u_1)^2 + \frac{1}{a_{22}} (u_2)^2 + \frac{1}{a_{33}} (u_3)^2 = 0.$$

6. The principal axes of a cone.

The equation of the plane conjugate to the direction λ^r with respect to the cone is

$$a_{mn} \lambda^m \lambda^n = 0.$$

This plane will be perpendicular to λ^r provided

$$a_{rm} \lambda^m = \theta \lambda_r = \theta g_{rm} \lambda^m. \quad \dots \dots \dots (19)$$

That this equation may be satisfied θ must be a root of the determinantal equation

$$|a_{mn} - \theta g_{mn}| = 0. \quad \dots \dots \dots (20)$$

Conversely, if we can find a root θ of this equation we can then find a direction λ^r which will satisfy (19). Such a direction is

called a *principal direction* or *principal axis of the cone*. We shall show that *there are three principal axes which are mutually orthogonal* and we shall find the form of the equation of the cone when referred to its principal directions as coordinate axes.

We first observe that the equation (20) has three roots and, since $g_{mn} x^m x^n$ is positive definite, we know (p. 16) that these roots are all real. Let us denote them by $\theta_1, \theta_2, \theta_3$. Two or more of these roots may of course be equal. Corresponding to the root θ_1 we have a direction $\lambda'_{(1)}$ satisfying (19), that is,

$$a_{rm} \lambda'_{(1)} = \theta_1 g_{rm} \lambda'_{(1)} \dots \dots \dots (21)$$

Let us now choose a rectangular Cartesian system x'^r in which $\lambda'_{(1)}$ is the x'^1 -axis, and in this system we have

$$\lambda'_{(1)} = \delta'_1, \quad g'_{11} = g'_{22} = g'_{33} = 1, \quad g'_{12} = g'_{23} = g'_{31} = 0.$$

We deduce from (21), which are vector equations,

$$a'_{r1} = \theta_1 g'_{r1},$$

or,

$$a'_{11} = \theta_1, \quad a'_{12} = a'_{13} = 0.$$

Our cone becomes

$$\theta_1 (x'^1)^2 + a'_{22} (x'^2)^2 + 2 a'_{23} x'^2 x'^3 + a'_{33} (x'^3)^2 = 0.$$

If $a'_{22} = a'_{23} = a'_{33} = 0$, we proceed no further, but if these relations are not true we take the section of the cone by the plane $x'^1 = 0$. It is

$$a'_{22} (x'^2)^2 + 2 a'_{23} x'^2 x'^3 + a'_{33} (x'^3)^2 = 0. \dots \dots (22)$$

This is the equation of two straight lines and we may take as coordinate axes the bisectors of these lines, the coordinate system being still rectangular Cartesian. Referred to these final coordinates \bar{x}^r the lines (22) become

$$\bar{a}_{22} (\bar{x}^2)^2 + \bar{a}_{33} (\bar{x}^3)^2 = 0.$$

In other words, the tensor \bar{a}_{mn} has all its components zero except $\bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}$. Now the left-hand side of (20) we know to be a relative invariant of weight 2; consequently if it vanishes in one system of coordinates it vanishes in every other. Thus in the system \bar{x}^r we have that the equation

$$\begin{array}{ccc} \bar{a}_{11} - \theta & 0 & 0 \\ 0 & \bar{a}_{22} - \theta & 0 \\ 0 & 0 & \bar{a}_{33} - \theta \end{array} = 0$$

has the same roots as (20). But the roots of this last equation are $\bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}$, that is,

$$\bar{a}_{11} = \theta_1, \quad \bar{a}_{22} = \theta_2, \quad \bar{a}_{33} = \theta_3,$$

and the equation of the cone is

$$\theta_1 (\bar{x}^1)^2 + \theta_2 (\bar{x}^2)^2 + \theta_3 (\bar{x}^3)^2 = 0, \quad . . . \quad (23)$$

where $\theta_1, \theta_2, \theta_3$ are the roots of (20) and the coordinate system is rectangular Cartesian. We also see that the vector equations (19) are satisfied in the new system by the coordinate axes which are therefore the principal axes. Of course it may happen that some of the θ 's vanish.

We have proved that *we can always choose a rectangular Cartesian system such that in it any symmetric covariant tensor a_{mn} will have*

$$a_{mn} = 0, \quad m \neq n.$$

EXAMPLES.

1. If $\theta_2 = \theta_3$, that is, two of the roots of (20) are equal, prove that every direction perpendicular to $\lambda_{(1)}$ is a principal direction of the cone. In this case the cone is said to be of *revolution*, the direction $\lambda_{(1)}$ being called the axis of revolution.

2. If the three roots of (20) are equal, prove that every direction is a principal direction. In this case the cone is said to be *isotropic*, and we see that its equation may be written $g_{mn} x^m x^n = 0$.

3. State and prove the dual theorem of § 6.

[The dual theorem is that the conic $a^{mn} u_m u_n = 0$ can have its equation reduced to canonical form and at the same time the axes be chosen rectangular Cartesian.]

7. The classification of cones.

We have seen that the equation of a cone can be put in the form

$$\bar{a}_{11} (\bar{x}^1)^2 + \bar{a}_{22} (\bar{x}^2)^2 + \bar{a}_{33} (\bar{x}^3)^2 = 0,$$

and at the same time the axes can be chosen rectangular Cartesian. We shall now classify the types of cones that can exist. We shall use the usual notation

$$A = |a_{mn}|, \quad g = |g_{mn}|,$$

and denote by A^{mn} the cofactor of a_{mn} in A . Since A and g are relative invariants of weight 2 and A^{mn} is a relative tensor of weight 2, we see that A/g is an absolute invariant and A^{mn}/g is an absolute tensor. In our new coordinate system \bar{x} we have $\bar{g} = 1$ and

$$\bar{A} = \bar{a}_{11} \bar{a}_{22} \bar{a}_{33}, \quad \bar{A}^{11} = \bar{a}_{22} \bar{a}_{33}, \quad \bar{A}^{23} = 0, \text{ etc.}$$

Hence

$$\frac{A}{g} = \bar{a}_{11} \bar{a}_{22} \bar{a}_{33}, \quad \frac{g_{mn} A^{mn}}{g} = \bar{a}_{22} \bar{a}_{33} + \bar{a}_{33} \bar{a}_{11} + \bar{a}_{11} \bar{a}_{22}.$$

Also $g^{mn} a_{mn}$ is an invariant, that is,

$$g^{mn} a_{mn} = \bar{g}^{mn} \bar{a}_{mn} = \bar{a}_{11} + \bar{a}_{22} + \bar{a}_{33}.$$

We conclude that $\bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}$ are the roots of the cubic equation

$$\theta^3 - \theta^2 g^{mn} a_{mn} + \theta \frac{g_{mn} A^{mn}}{g} - \frac{A}{g} = 0, \quad \dots \quad (24)$$

or,

$$g \theta^3 - \theta^2 G^{mn} a_{mn} + \theta g_{mn} A^{mn} - A = 0, \quad \dots \quad (25)$$

where $G^{mn} = g g^{mn}$ is the cofactor of g_{mn} in g . Equation (25) is just the determinantal equation

$$|a_{mn} - \theta g_{mn}| = 0, \quad \dots \quad (26)$$

and $\bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}$ are the roots of this equation, a result which we obtained in the last section by another method of proof.

I. $A \neq 0$, then none of the roots is zero and the cone is non-degenerate. If the three roots are all positive or all negative the cone is imaginary, otherwise it is real.

II. $A = 0$, but the first minors of A are not all zero. All the components of the relative tensor A^{mn} are not zero and therefore they cannot be all zero in any other coordinate system. Since A is zero, one root of (25) at least is zero, say \bar{a}_{33} , and we see that the only component of \bar{A}^{mn} that survives is $\bar{A}^{33} = \bar{a}_{11} \bar{a}_{22}$. Consequently neither \bar{a}_{11} nor \bar{a}_{22} can vanish and there is only one zero root. The cone becomes

$$\bar{a}_{11} (\bar{x}^1)^2 + \bar{a}_{22} (\bar{x}^2)^2 = 0,$$

which is the equation of *two distinct planes* through the \bar{x}^3 -axis. The planes are imaginary if \bar{a}_{11} and \bar{a}_{22} have the same sign, otherwise they are real.

III. A , together with all its first minors, vanishes. Then $A^{mn} = 0$, and (25) has two zero roots, say \bar{a}_{11} and \bar{a}_{22} . The equation of the cone is

$$\bar{a}_{33} (\bar{x}^3)^2 = 0,$$

which is the equation of *two coincident planes* $\bar{x}^3 = 0$.

It will be noticed that the conditions stated for each case are *invariant conditions* which if true in one coordinate system are true in all systems.

EXAMPLE.

Show that we have the following classification of the conic $\alpha^{mn} u_m u_n = 0$.

- (i) $|\alpha^{mn}| \neq 0$, the conic is non-degenerate.
- (ii) $|\alpha^{mn}| = 0$, but all first minors are not zero, the conic reduces to two distinct points at infinity.
- (iii) $|\alpha^{mn}|$ and all its first minors vanish, the conic reduces to two coincident points at infinity.

EXAMPLES VI.

1. Show that the cone $(a_{mn} - \theta g_{mn}) x^m x^n = 0$ degenerates into two distinct or coincident planes according as θ equals a single or double root of the equation $|a_{mn} - \theta g_{mn}| = 0$.

2. If the cone $a_{mn} x^m x^n = 0$ is of revolution, show that we can find a quantity θ such that

$$A^{rs} - \theta g^{\epsilon^r m n} \epsilon^{s r q} a_{m p} g_{n q} + \theta^2 g^r s = 0.$$

[The cone $(a_{mn} - \theta g_{mn}) x^m x^n = 0$ degenerates into two coincident planes when θ equals the double root of $|a_{mn} - \theta g_{mn}| = 0$. Hence all the first minors of the determinant vanish for this value of θ .]

3. Show that in rectangular Cartesian coordinates the conditions of Ex. 2 become

$$a_{11} a_{23} - a_{12} a_{13} = \theta a_{23}, \text{ etc.}$$

4. Show that if a given plane cuts the cone $a_{mn} x^m x^n = 0$ in two distinct generators we can choose a coordinate system in which $\bar{x}^3 = 0$ is the given plane and the equation of the cone becomes

$$\bar{a}_{33} (\bar{x}^3)^2 + 2 \bar{a}_{12} \bar{x}^1 \bar{x}^2 = 0.$$

5. Prove that the angle α between the lines in which the plane $u_r x^r = 0$ cuts the cone $a_{mn} x^m x^n = 0$ is given by

$$\cot^2 \alpha = - \frac{A \{ g_{mn} (\alpha^{m r} \alpha^{r s} - \alpha^{m r} \alpha^{n s}) u_r u_s \}^2}{4 g^{mn} u_m u_n \alpha^{r s} u_r u_s}.$$

[Evaluate the invariant in the special coordinate system of Ex. 4.]

6. *Isotropic cone.* The cone $g_{mn} x^m x^n = 0$ is called the isotropic cone. Show that if two directions at O are conjugate with respect to the isotropic cone they are orthogonal and conversely.

7. *Circle at infinity.* The conic $g^{mn} u_m u_n = 0$ is called the circle at infinity. Show that the isotropic cone is the cone obtained by joining O to the points of the circle, and show that two planes are orthogonal if they are conjugate with respect to the circle at infinity.

8. *Reciprocal cones.* Show that the lines through O perpendicular to the tangent planes of $a_{mn} x^m x^n = 0$ lie on the cone

$$g_{mr} g_{ns} \alpha^{rs} x^m x^n = 0,$$

and show that the perpendiculars to the tangent planes of the latter cone lie on the original cone. Two such cones are called reciprocal cones.

9. If $a_{mn} x^m x^n = 0$ is expressed in canonical form, the axes being orthogonal Cartesian, show that the reciprocal cone is

$$-\frac{(\bar{x}^1)^2}{a_{11}} + \frac{(\bar{x}^2)^2}{\bar{a}_{22}} + \frac{(\bar{x}^3)^2}{\bar{a}_{33}} = 0.$$

10. The vanishing of the invariant $g^{mn} a_{mn}$ means that there is an infinity of triads of mutually orthogonal generators of $a_{mn} x^m x^n = 0$.

11. The vanishing of the invariant $g_{mn} \alpha^{mn}$ means that there is an infinity of triads of mutually orthogonal tangent planes to $a_{mn} x^m x^n = 0$.

having the origin O as a common vertex, then any cone with the same vertex and passing through the points common to (1) is of the form

$$(a_{mn} - \theta b_{mn}) x^m x^n = 0. \quad \dots \quad (2)$$

Therefore as θ varies we get a system of cones all of which contain the common points of the two original cones (1).

The system (2) will contain members which are not proper cones but are pairs of planes, namely those members for which θ is a root of the equation

$$|a_{mn} - \theta b_{mn}| = 0. \quad \dots \quad (3)$$

This is a cubic equation and has three roots; therefore in general the family (2) has three pairs of planes belonging to it. We shall often refer to these members as *the degenerate members of the family*.

If we transform to a new coordinate system \bar{x}^r the system becomes

$$(a_{mn} - \theta b_{mn}) \bar{x}^m \bar{x}^n = 0, \quad (4)$$

the parameter θ giving the same member in the new variables as it gave in the old. Also the determinantal equation (3) becomes

$$-\theta \bar{b}_{mn} = 0 \quad (5)$$

Now the determinant is known to be a relative invariant of weight 2. Hence if it vanishes for any value of θ in one coordinate system, it will vanish for that value in every other system. In other words, the roots of the equation (5) are the same as the roots of (3), that is, *the roots of (3) are invariants*.

Our object is to examine the different ways in which the two cones (1) may be related to each other and we can do this by examining the different forms that the system (2) can assume.

EXAMPLE.

Show that the family of conics $(\alpha^{mn} - \theta \beta^{mn}) u_m u_n = 0$ possesses degenerate members for the roots of $|\alpha^{mn} - \theta \beta^{mn}| = 0$, and show that these degenerate members are pairs of points at infinity.

2. The common polar directions of a family of cones.

The planes conjugate to the direction λ^r with respect to the two cones are respectively

$$\begin{aligned} a_{mn} x^m \lambda^n &= 0, \\ b_{mn} x^m \lambda^n &= 0. \end{aligned}$$

These planes will coincide if

$$a_{mn} \lambda^n = \theta b_{mn} \lambda^n, \quad (6)$$

and we see that if this condition is satisfied then the planes conjugate to λ^r with respect to all the cones of the family coincide. We shall call such a direction a *common polar direction of the family*. If (6) is true, it follows that θ must be a root of the equation (3) and, conversely, if θ is a root of (3) we can always find a corresponding direction λ^r satisfying (6). There is therefore always at least one common polar direction.

If λ_0^r is a common polar direction of the family we have seen that there is a number θ_0 such that

$$(a_{mn} - \theta_0 b_{mn}) \lambda_0^n = 0,$$

and θ_0 is a root of (3). Hence the cone

$$(a_{mn} - \theta_0 b_{mn}) x^m x^n = 0$$

reduces to a pair of planes which intersect in the line λ_0^r . Moreover if λ_0^r is a generator of the cone $b_{mn} x^m x^n = 0$, we see that

$$a_{mn} \lambda_0^m \lambda_0^n = \theta_0 b_{mn} \lambda_0^m \lambda_0^n = 0,$$

and λ_0^r is a generator of every member of the family. But the plane conjugate to a generator λ_0^r is the tangent plane along this generator. Consequently all the cones have a common tangent plane, and we conclude that if a common polar direction is a generator of one non-degenerate member of the family then all the members touch along this generator.

If there are *two* independent common polar directions λ_0^r and λ_1^r , we have

$$a_{mn} \lambda_0^n = \theta_0 b_{mn} \lambda_0^n, \quad a_{mn} \lambda_1^n = \theta_1 b_{mn} \lambda_1^n.$$

Now θ_0 may or may not equal θ_1 , and we consider the two cases:

(i) $\theta_0 \neq \theta_1$. We find that

$$a_{mn} \lambda_1^m \lambda_0^n = \theta_0 b_{mn} \lambda_1^m \lambda_0^n = \theta_1 b_{mn} \lambda_0^m \lambda_1^n,$$

and consequently

$$a_{mn} \lambda_1^m \lambda_0^n = b_{mn} \lambda_1^m \lambda_0^n = 0.$$

The directions λ_0^r and λ_1^r are therefore conjugate with respect to all the members of the family.

(ii) $\theta_0 = \theta_1$. Here we see that

$$a_{mn} (\alpha \lambda_0^n + \beta \lambda_1^n) = \theta_0 b_{mn} (\alpha \lambda_0^n + \beta \lambda_1^n),$$

and every direction in the plane containing λ_0'' and λ_1'' is a common polar direction. Let us choose any two vectors in this plane which are conjugate with respect to the cone $b_{mn} x^m x^n = 0$. Then these two vectors are conjugate with respect to any member of the family. *Therefore if we have two independent common polar directions of the family we can always choose two such directions which are conjugate with respect to every member of the system.*

If we have three independent common polar directions $\lambda_0'', \lambda_1'', \lambda_2''$, we have

$$a_{mn} \lambda_0'' = \theta_0 b_{mn} \lambda_0'', \quad a_{mn} \lambda_1'' = \theta_1 b_{mn} \lambda_1'', \quad a_{mn} \lambda_2'' = \theta_2 b_{mn} \lambda_2''.$$

We mean by three independent directions three which are not coplanar. We have the following cases:

(i) No two of the θ 's are equal. We see exactly as in (i) of the last paragraph that the directions $\lambda_0'', \lambda_1'', \lambda_2''$ are all mutually conjugate with respect to every member of the family.

(ii) Two of the θ 's are equal, say $\theta_0 = \theta_1$. We see that every direction in the plane containing λ_0'' and λ_1'' is a common polar direction and we can choose two in the plane which are also conjugate with respect to every member. Moreover λ_2'' is conjugate to every direction in the plane of λ_0'' and λ_1'' .

(iii) All three of the θ 's are equal. In this case every direction is a common polar direction and we have only to take three directions which will be mutually conjugate with respect to $b_{mn} x^m x^n = 0$ and we see that they will also be mutually conjugate with respect to every member of the system.

We conclude that if we have three independent common polar directions we can always choose three such directions which are also mutually conjugate with respect to every member of the family.

EXAMPLES.

1. If $b_{mn} = g_{mn}$ show that there are always three common polar directions which are orthogonal to each other.

2. Show that if (3) has two distinct roots then there must be two independent common polar directions.

[If λ_0'' and λ_1'' are the two directions corresponding to the unequal roots θ_0 and θ_1 , then $\lambda_1'' \neq k \lambda_0''$.]

3. If equation (3) has three distinct roots show that there must be three independent common polar directions.

4. Investigate the dual theorems regarding a system of conics $(\alpha^m n - \theta \beta^m n) u_m u_n = 0$, showing that to a common polar direction there corresponds a common polar line at infinity, that is, a line at infinity which has the same pole with respect to all the conics.

3. The canonical forms of the equation of a family of cones.

We shall now investigate the simplest forms to which we can reduce the equation (2), p. 89, of the family of cones and in so doing we shall examine the different relations that the two cones (1) can have to each other. We may call the cones

$$a_{mn} x^m x^n = 0, \quad b_{mn} x^m x^n = 0$$

the *base cones*, and we shall leave aside the singular case where all the cones of the family are degenerate, so that we can always choose our base cones to be proper cones. In other words we choose our base cones such that $|a_{mn}|$ and $|b_{mn}|$ are different from zero.

We shall divide the discussion into different sections according to the number of independent common polar directions that exist.

I. *Three independent common polar directions.* We have seen that these three directions can be chosen to be mutually conjugate with respect to both base cones and consequently with respect to every cone of the family. Let us take these directions as the coordinate axes. Since the axes are conjugate with respect to both cones we have in our new coordinate system

$$\bar{a}_{23} = \bar{a}_{31} = \bar{a}_{12} = \bar{b}_{23} = \bar{b}_{31} = \bar{b}_{12} = 0,$$

and the cones become

$$\begin{aligned} \bar{a}_{11} (\bar{x}^1)^2 + \bar{a}_{22} (\bar{x}^2)^2 + \bar{a}_{33} (\bar{x}^3)^2 &= 0, \\ \bar{b}_{11} (\bar{x}^1)^2 + \bar{b}_{22} (\bar{x}^2)^2 + \bar{b}_{33} (\bar{x}^3)^2 &= 0. \end{aligned}$$

The determinant $\iota_{mn} - \theta b_{mn}$ is now

$$\begin{vmatrix} \bar{a}_{11} - \theta \bar{b}_{11} & 0 & 0 \\ 0 & \bar{a}_{22} - \theta \bar{b}_{22} & 0 \\ 0 & 0 & \bar{a}_{33} - \theta \bar{b}_{33} \end{vmatrix} \\ = (\bar{a}_{11} - \theta \bar{b}_{11}) (\bar{a}_{22} - \theta \bar{b}_{22}) (\bar{a}_{33} - \theta \bar{b}_{33}).$$

Hence the roots of the determinantal equation (3), p. 89, are

$$\frac{\bar{a}_{11}}{\bar{b}_{11}}, \quad \frac{\bar{a}_{22}}{\bar{b}_{22}}, \quad \frac{\bar{a}_{33}}{\bar{b}_{33}}.$$

Denoting these roots by θ_1 , θ_2 and θ_3 respectively we therefore see that the equation of the family is

$$\bar{b}_{11}(\theta_1 - \theta)(\bar{x}^1)^2 + \bar{b}_{22}(\theta_2 - \theta)(\bar{x}^2)^2 + \bar{b}_{33}(\theta_3 - \theta)(\bar{x}^3)^2 = 0, \quad (7)$$

and the determinant becomes

$$\begin{vmatrix} \theta_1 - \theta & 0 & 0 \\ \bar{b}_{11} & \bar{b}_{22} & \bar{b}_{33} \\ 0 & 0 & 0 \end{vmatrix} \quad \dots \quad (8)$$

There are three sub-cases:

(A) $\theta_1, \theta_2, \theta_3$ are unequal. The two base cones intersect in *four distinct generators* and the degenerate members of the family are the three pairs of planes given for $\theta = \theta_1, \theta_2, \theta_3$. To illustrate the cases we shall show the section of the family by a plane not passing through O . The common generators are OA, OB, OC, OD (fig. 1). The degenerate members are the pairs of planes $OAB, OCD; OBC, ODA; OBD, OCA$, and the three common polar directions are their lines of intersection, namely, OP, OQ, OR . If we write any one of the system $S = 0$, and if $P = 0, Q = 0$ is any one of the above pairs of planes, then the family of cones is of the form

$$S + \lambda PQ = 0.$$

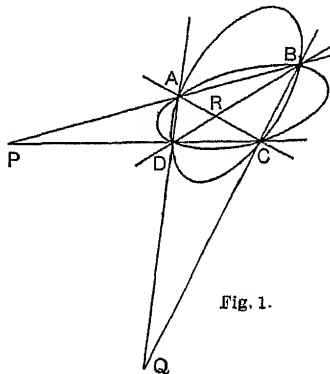


Fig. 1.

(B) *Two roots equal*, ($\theta_1 = \theta_2 \neq \theta_3$). The cones have *double contact* along the two generators in which $\bar{x}^3 = 0$ cuts the cones, that is, the generators OA and OB (fig. 2).

The degenerate members are a pair of coincident planes OAB , given by $\theta = \theta_1 = \theta_2$, and the pair of tangent planes OPA, OPB . The common polar directions are OP, OQ, OR , where Q and R are any two points on AB which are conjugate with respect to the cones. If we denote the equation of OAB by $P = 0$, the family may be written

$$S + \lambda P^2 = 0.$$

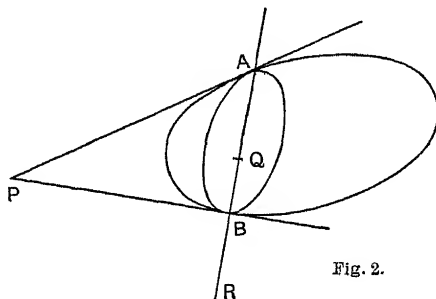


Fig. 2.

(C) *All three roots equal*, ($\theta_1 = \theta_2 = \theta_3$). All the cones now coincide for the two base cones are identical. The form of the family is merely

$$S = 0.$$

II. *Two independent polar directions.* Let these two directions be OP and OQ . Now OP and OQ have each the same polar planes with respect to every member of the family and they can be chosen so as to be conjugate with respect to every member. Also the polar plane of either OP or OQ must coincide with the plane OPQ as otherwise the two polar planes would intersect in a direction OR , which would be a common polar direction and would not be coplanar with OP and OQ . Let OP be the one whose polar plane is OPQ . Consequently OP is a generator of every member of the family and OPQ is the common tangent plane to every member along OP . In other words, all the cones touch along OP . Moreover the polar plane of OQ cannot coincide with OPQ . Let it cut the cone B in OP and OR . We shall take OP as the \bar{x}^1 -axis, OR the \bar{x}^2 -axis and OQ the \bar{x}^3 -axis. Since OP is a generator and OP , OQ are conjugate with respect to both base conics,

$$\bar{a}_{11} = \bar{b}_{11} = \bar{a}_{13} = \bar{b}_{13} = 0.$$

Also, since OR is a generator of B and OQ , OR are conjugate with respect to A and B ,

$$\bar{b}_{22} = \bar{b}_{23} = \bar{a}_{23} = 0.$$

Hence the determinant becomes

$$\begin{vmatrix} 0 & , & \bar{a}_{12} - \theta \bar{b}_{12}, & 0 \\ \bar{a}_{21} - \theta \bar{b}_{21}, & & \bar{a}_{22} & , & 0 \\ 0 & , & 0 & , & \bar{a}_{33} - \theta \bar{b}_{33} \end{vmatrix} \\ = - (\bar{a}_{33} - \theta \bar{b}_{33}) (\bar{a}_{12} - \theta \bar{b}_{12})^2.$$

We see that two of the roots of the determinantal equation are equal. Let the roots be θ_1 , θ_1 , θ_3 . Then the family becomes

$$2 \bar{b}_{12} (\theta_1 - \theta) \bar{x}^1 \bar{x}^2 + \bar{a}_{22} (\bar{x}^2)^2 + \bar{b}_{33} (\theta_3 - \theta) (\bar{x}^3)^2 = 0, \quad (9)$$

and the determinant may be written

$$\bar{b}_{12}^2 \bar{b}_{33} \begin{vmatrix} 0 & , & \theta_1 - \theta, & 0 \\ \theta_1 - \theta, & & \bar{a}_{22} & , & 0 \\ 0 & , & 0 & , & \theta_3 - \theta \end{vmatrix} \dots \dots (10)$$

We consider the two sub-cases:

(A) $\theta_1 \neq \theta_3$. The determinantal equation has only two equal roots. The cones have *single contact* along OP (fig. 3) and there are two other distinct generators OA, OB . The degenerate members are the planes OPQ, OQA , given by $\theta = \theta_3$, and the planes OPB, OPB , given by $\theta = \theta_1$. The two common polar directions are OP and OQ . If we write the equation of OPQ as $P = 0$ and that of OAB as $Q = 0$, the family of cones may be written

$$S + \lambda PQ = 0,$$

where $P = 0$ is a tangent plane to all the cones.

(B) $\theta_1 = \theta_3$. The determinantal equation has three equal roots. The cones have *four-point contact* along OP (fig. 4) and there are no other generators common. The only degenerate member is the plane OPQ , taken twice, which is given by $\theta = \theta_1$. The common polar directions are OP and OQ , Q being any point on the line PQ . If we write the equation of OPQ as $P = 0$, the family of cones become

$$S + \lambda P^2 = 0,$$

where $P = 0$ is a tangent plane to all the cones.

III. *One common polar direction.* Let OP be this direction. The polar plane of OP must pass through OP , otherwise this plane would cut the base cones in two pairs of distinct lines and we could find at least one other direction in this plane which would be a common polar direction. Hence OP is a generator and the common polar plane is the tangent plane along OP to all the cones, that is, the cones touch along OP . Let OQ be any direction in the polar plane of OP and let OR be the intersection of the cone B by the polar plane of OQ with respect to B . We shall take OP as the \bar{x}^1 -axis, OQ as the \bar{x}^3 -axis and OR the \bar{x}^2 -axis. Since OP is a generator of both cones and OP, OQ are conjugate with respect to both cones,

$$a_{11} = b_{11} = \bar{a}_{13} = \bar{b}_{13} = 0.$$

Since OR is a generator of the cone B and OQ, OR are conjugate with respect to B ,

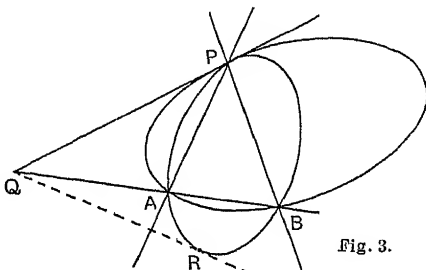


Fig. 3.

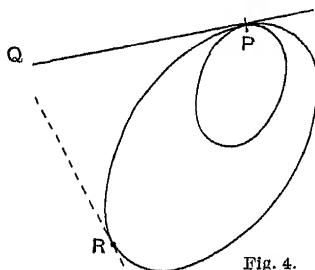


Fig. 4.

$$\bar{b}_{22} = \bar{b}_{23} = 0.$$

Hence the determinant is

$$\begin{vmatrix} 0 & , & \bar{a}_{12} - \theta \bar{b}_{12}, & 0 \\ \bar{a}_{21} - \theta \bar{b}_{21}, & \bar{a}_{22} & , & \bar{a}_{23} \\ 0 & , & \bar{a}_{32} & , & \bar{a}_{33} - \theta \bar{b}_{33} \end{vmatrix} \\ = -(\bar{a}_{21} - \theta \bar{b}_{21})(\bar{a}_{32} - \theta \bar{b}_{32})(\bar{a}_{12} - \theta \bar{b}_{12})^2.$$

Now there can be only one distinct root of the determinantal equation since there is only one common polar direction. Hence if we denote this root by θ , we must have

$$\bar{a}_{12} = \theta_1 \bar{b}_{12}, \quad \bar{a}_{33} = \theta_1 \bar{b}_{33}.$$

The family of cones is therefore

$$2 \bar{b}_{12}(\theta_1 - \theta) \bar{x}^1 \bar{x}^2 + 2 \bar{a}_{23} \bar{x}^2 \bar{x}^3 + \bar{a}_{22}(\bar{x}^2)^2 \\ + \bar{b}_{33}(\theta_1 - \theta) (\bar{x}^3)^2 = 0,$$

and the degenerate member is given by $\theta = \theta_1$, that is, by

$$\bar{x}^2 = 0, \quad 2 \bar{a}_{23} \bar{x}^3 + \bar{a}_{22} \bar{x}^2 = 0.$$

The second plane cuts the cones in OP and another generator different from OP , that is, there is a common generator other than OP . We may simplify our results by taking this generator as OR , the \bar{x}^2 -axis, in which case

$$\bar{a}_{22} = 0.$$

The system of cones may then be written

$$2 \bar{b}_{12}(\theta_1 - \theta) \bar{x}^1 \bar{x}^2 + 2 \bar{a}_{23} \bar{x}^2 \bar{x}^3 + \bar{b}_{33}(\theta_1 - \theta) (\bar{x}^3)^2 = 0, \quad (11)$$

and the determinant becomes

$$\begin{vmatrix} 0 & , & \theta_1 - \theta, & 0 \\ \bar{b}_{33} \bar{b}_{12} & & 0 & \bar{a}_{23} \\ 0 & & (\theta_1 - \theta) & \end{vmatrix} \dots \dots (12)$$

The cones have *three-point contact* along OP (fig. 5) and there is one other common generator OR . The only degenerate member is the pair of planes OPQ , OPR , given by $\theta = \theta_1$. The only common polar

direction is OP . If we write the equations of the planes OPQ , OPR as $P=0$, $Q=0$ respectively, we see that the family of cones is

$$S + \lambda PQ = 0,$$

where $P=0$ is a tangent plane to all the cones and the plane $Q=0$ passes through the generator of contact.

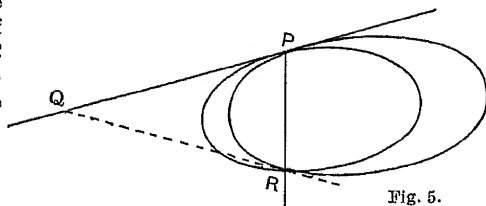


Fig. 5.

We have exhausted all the possible types that can arise and our next problem will be to discriminate analytically between the different cases.

EXAMPLES.

1. Show that \bar{a}_{22} cannot vanish in equation (9).
[Its vanishing would make OR a common polar direction.]
2. Show that \bar{a}_{23} cannot vanish in equation (11).
3. Prove the following dual results for the system of conics

$$(\alpha^{mn} - \theta \beta^{mn}) u_m u_n = 0,$$

the algebraic work being practically identical with that of § 3 (p. 92):

- I (A). The conics all touch four distinct lines at infinity.
- I (B). The conics touch two distinct lines at infinity in two given points (double contact).
- I (C). The conics are identical.
- II (A). The conics touch a straight line at a given point (single contact) and touch two other distinct straight lines.
- II (B). The conics all touch a straight line at a given point (contact of the third order) and have no other tangent lines in common.
- III. The conics touch a given straight line at a given point (contact of the second order) and have one other common tangent line.

[The canonical forms of the different cases are exactly similar to those above and we have merely to interpret our results by the principle of duality.]

4. The theory of elementary divisors.

In order that we may discriminate completely between the different cases of § 3 (p. 92) we shall have to digress a little into the theory of the elementary divisors of a determinant.

Let a_{mn} and b_{mn} be two covariant tensors with respect to linear transformations, and let us put

$$h_{mn} \equiv a_{mn} - \theta b_{mn}. \quad \dots \dots (13)$$

We wish to consider the determinant

$$|h_{mn}| \equiv |a_{mn} - \theta b_{mn}| \dots \dots \dots (14)$$

We know that $|h_{mn}|$ is a relative invariant of weight 2, that its cofactors H^{mn} form a relative tensor of weight 2 and that its elements h_{mn} form an absolute tensor. Thus if the variables are given a transformation

$$\bar{x}^r = c_s^r x^s, \quad x^r = \gamma_s^r \bar{x}^s \quad c = \quad \gamma = |\gamma_s^r|$$

we have the formulae

$$\begin{aligned} |\bar{h}_{mn}| &= \gamma^2 |h_{mn}|, \\ \bar{H}^{mn} &= \gamma^2 c_p^m c_q^n H^{pq}, \quad H^{mn} = c^2 \gamma_p^m \gamma_q^n \bar{H}^{pq}, \quad (15) \\ \bar{h}_n &- \gamma_m^p \gamma_n^q h_{pq}, \quad h_{mn} = c_m^p c_n^q \bar{h}_{pq}. \end{aligned}$$

We shall assume that the determinant $|b_{mn}|$ is not zero.

Let us take the *highest common factor* of *all* the elements h_{mn} and let us denote it by $D_1(\theta)$. We see that it cannot be of higher degree in θ than 1, and we shall choose it so that the first coefficient is always unity, that is, $D_1(\theta)$ will either be 1 or $(\theta - \alpha)$. In the same way let us take the *highest common factor* of *all* the two-rowed determinants in (14), or, what is the same thing, the highest common factor of all the cofactors H^{mn} . We shall denote it by $D_2(\theta)$, and again take the first coefficient as unity, so that $D_2(\theta)$ is one of the three forms 1, $(\theta - \alpha)$, $(\theta - \alpha)(\theta - \beta)$. Lastly we take the determinant itself and divide by $-|b_{mn}|$, and we shall denote the result by $D_3(\theta)$ so that $D_3(\theta)$ is a cubic polynomial in θ whose first coefficient is unity.

We see from (15) that, since the c 's are constants, the highest common factor of *all* the h_{mn} is also the highest common factor of *all* the \bar{h}_{mn} and vice versa. Hence $D_1(\theta)$ is an invariant under the linear transformation and similarly $D_2(\theta)$ and $D_3(\theta)$ are invariants. Moreover a determinant can be linearly expanded in terms of the cofactors. Therefore $D_3(\theta)$ must contain $D_2(\theta)$ as a factor and in the same way $D_2(\theta)$ must contain $D_1(\theta)$. We can then write

$$E_3(\theta) = \frac{D_3(\theta)}{D_2(\theta)}, \quad E_2(\theta) = \frac{D_2(\theta)}{D_1(\theta)}, \quad E_1(\theta) = \frac{D_1(\theta)}{1} \quad (16)$$

where the E 's are polynomials in θ , and we see that *they are all invariants*. These polynomials E_i are often called the *invariant factors of the determinant* (14).

We shall now factorise each of the invariant factors. Let $(\theta - \alpha)$ be a linear factor of any one, and let this factor be repeated e_1 times in E_1 , e_2 times in E_2 and e_3 times in E_3 . Then such of the expressions

$$(\theta - \alpha)^{e_1}, \quad (\theta - \alpha)^{e_2}, \quad (\theta - \alpha)^{e_3} \quad . \quad . \quad . \quad (17)$$

as are not merely constants are called the elementary divisors of (14) corresponding to the linear factor $(\theta - \alpha)$. We therefore see that we have elementary divisors corresponding to each of the linear factors of the E 's. Moreover since the E 's are invariants we infer that the elementary divisors are invariants and, in particular, the numbers e_1, e_2, e_3 are invariants.

It is the indices that are of importance in (17) and we shall often express the fact that (17) are the elementary divisors corresponding to the linear factor $(\theta - \alpha)$ by the symbol

$$(e_3 e_2 e_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

enclosing in round brackets the indices belonging to any one factor $(\theta - \alpha)$. The symbol (18) is called the *characteristic of the factor*. Also we deduce from (16) that

$$D_3(\theta) = E_1(\theta) E_2(\theta) E_3(\theta), \quad . \quad . \quad . \quad . \quad (19)$$

which shows that any factor of the E 's is also a factor of D_3 and hence is a factor of the determinant (14). Consequently when examining (14) for elementary divisors we need consider only the linear factors of the determinant.

As a simple example let us consider the determinant

$$\begin{array}{ccc} \theta - \alpha & 0 & 0 \\ 0 & \theta - \alpha & 0 \\ 0 & 0 & \theta - \beta \end{array} = (\theta - \alpha)^2 (\theta - \beta).$$

Here $D_3 = (\theta - \alpha)^2 (\theta - \beta)$, $D_2 = (\theta - \alpha)$, $D_1 = 1$ and so $E_3 = (\theta - \alpha)$, $E_2 = (\theta - \alpha)$, $E_1 = 1$. Corresponding to the factor $(\theta - \alpha)$ the symbol is therefore (1 1 0); in practice we leave out the zeros and read this (11). Corresponding to $(\theta - \beta)$ the symbol is (1 0 0), or, leaving out the zeros, it becomes 1 since no bracket is now required. The whole information about the elementary divisors is then contained in the characteristic [(11)1], where indices inside a round bracket apply to one factor.

We shall find that we can classify the different cases of § 3 (p. 92) by means of the elementary divisors of (14).

EXAMPLES.

1. Show that D_1^2 is a factor of D_2 and D_1^3 a factor of D_3 , and deduce that E_2 contains E_1 as a factor.

2. Show that $D_3 D_1$ contains D_2^2 as a factor and deduce that E_3 contains E_2 as a factor.

[If K_{mn} is the cofactor of H^{mn} in the reciprocal determinant, then $K_{mn} = |h_{rs}| h_{mn}$. But K_{mn} is divisible by D_2^2 and $D_1 D_3$ is the highest common factor of $|h_{rs}| h_{mn}$, etc.]

3. Infer from Exs. 1 and 2 that, in (18), $e_3 > e_2 > e_1$.

5. Analytical discrimination of the cases.

Let us consider the different cases of § 3 (p. 92), examining the elementary divisors of the determinant (14) in each case. Since the elementary divisors are invariants it is sufficient to find them in any coordinate system.

I(A). The determinant for this particular case is, in its simplest form, (8), p. 93, where $\theta_1, \theta_2, \theta_3$ are all different. We have

$$\begin{aligned} D_3(\theta) &= (\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3), & D_2(\theta) &= 1, & D_1(\theta) &= 1, \\ E_3(\theta) &= (\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3), & E_2(\theta) &= 1, & E_1(\theta) &= 1. \end{aligned}$$

There are three distinct roots and the characteristic symbol for this case is

$$[111]. \quad \dots \dots \dots (20)$$

I(B). The determinant is (8), p. 93, where $\theta_1 = \theta_2$. Hence

$$\begin{aligned} D_3(\theta) &= (\theta - \theta_1)^2(\theta - \theta_3), & D_2(\theta) &= (\theta - \theta_1), & D_1(\theta) &= 1, \\ E_3(\theta) &= (\theta - \theta_1)(\theta - \theta_3), & E_2(\theta) &= (\theta - \theta_1), & E_1(\theta) &= 1. \end{aligned}$$

There are two distinct roots and the characteristic is

$$[(11)1]. \quad \dots \dots \dots (21)$$

I(C). The determinant is (8), p. 93, where $\theta_1 = \theta_2 = \theta_3$. Hence

$$\begin{aligned} D_3(\theta) &= (\theta - \theta_1)^3, & D_2(\theta) &= (\theta - \theta_1)^2, & D_1(\theta) &= (\theta - \theta_1), \\ E_3(\theta) &= (\theta - \theta_1), & E_2(\theta) &= (\theta - \theta_1), & E_1(\theta) &= (\theta - \theta_1). \end{aligned}$$

There is only one distinct root and the characteristic is

$$[(111)]. \quad \dots \dots \dots (22)$$

II(A). The determinant is here (10), p. 94, where θ_1 and θ_3 are not equal and \bar{a}_{22} is not zero.

$$\begin{aligned} D_3(\theta) &= (\theta - \theta_1)^2(\theta - \theta_3), & D_2(\theta) &= 1, & D_1(\theta) &= 1, \\ E_3(\theta) &= (\theta - \theta_1)^2(\theta - \theta_3), & E_2(\theta) &= 1, & E_1(\theta) &= 1. \end{aligned}$$

The number of distinct roots is two and we find for the characteristic

$$[21]. \dots \dots \dots (23)$$

II (B). The determinant is still (10), p. 94, but now $\theta_1 = \theta_3$.

$$\begin{aligned} D_3(\theta) &= (\theta - \theta_1)^3, & D_2(\theta) &= (\theta - \theta_1), & D_1(\theta) &= 1, \\ E_3(\theta) &= (\theta - \theta_1)^2, & E_2(\theta) &= (\theta - \theta_1), & E_1(\theta) &= 1. \end{aligned}$$

There is only one distinct root and we have for the characteristic

$$[(21)]. \dots \dots \dots (24)$$

III. In this case the determinant is (12), p. 96, in which \bar{a}_{23} is not zero. Thus

$$\begin{aligned} D_3(\theta) &= (\theta - \theta_1)^3, & D_2(\theta) &= 1, & D_1(\theta) &= 1, \\ E_3(\theta) &= (\theta - \theta_1)^3, & E_2(\theta) &= 1, & E_1(\theta) &= 1. \end{aligned}$$

There is only one distinct root and the characteristic reads

$$[3]. \dots \dots \dots (25)$$

We observe that the characteristic is different in each case and therefore serves to distinguish one case from another. Thus if we know the elementary divisors of

$$\theta b, \quad (26)$$

we know the canonical forms to which the quadratic forms

$$a_{mn} x^m x^n, \quad b_{mn} x^m x^n$$

can be reduced, namely, the particular case of those mentioned above whose characteristic is the same as that of (26).

Consequently if two pairs of quadratic forms $a_{mn} x^m x^n$, $b_{mn} x^m x^n$ and $a'_{mn} x'^m x'^n$, $b'_{mn} x'^m x'^n$ have the same elementary divisors the first pair can be transformed into the second by a linear transformation and vice versa. For the family

$$(a_{mn} - \theta b_{mn}) x^m x^n$$

can be reduced to its canonical form by a transformation T , and the family

$$(a'_{mn} - \theta b'_{mn}) x'^m x'^n$$

can be reduced to the same form by a transformation T' . Therefore, denoting inverse transformations by T^{-1} and T'^{-1} , we see that the transformation $T \cdot T'^{-1}$ will transform the first pair into the second, and $T' \cdot T^{-1}$ will transform the second pair into the first.

It must be remarked that we have excluded from the discussion the singular case where both the quadratic forms are degenerate, which is the case where the determinantal equation vanishes for all values of θ .

EXAMPLES.

1. Show that the coefficients of the cubic polynomial $|a_{mn} - \theta b_{mn}|$ are all relative invariants of weight 2.

We shall let A, B be the determinants $|a_{mn}|, |b_{mn}|$ respectively, A^{mn} the cofactor of a_{mn} in A and B^{mn} the cofactor of b_{mn} in B . When we expand we find that

$$|a_{mn} - \theta b_{mn}| = A - A^{mn} b_{mn} \theta + a_{mn} B^{mn} \theta^2 - B \theta^3.$$

Now A, B are relative invariants of weight 2 and A^{mn}, B^{mn} are relative tensors of the same weight. Hence the theorem follows immediately. These coefficients are often denoted by special symbols, the usual notation being $\mathcal{A}, \mathcal{O}, \mathcal{O}', \mathcal{A}'$. Since they are invariants we should expect that their vanishing will imply some geometrical relationship between the cones. Thus $\mathcal{A} = 0$ implies that the first cone degenerates, and $B = 0$ implies that the second degenerates. We shall find that the vanishing of the other two can also be interpreted geometrically.

2. Show that if $A^{mn} b_{mn} = 0$ there is an infinity of triads of generators of the B cone which are mutually conjugate with respect to the A cone. Similarly for $a_{mn} B^{mn} = 0$.

[Reduce A to canonical form and at the same time take two coordinate axes as generators of B , then calculate the invariant in this system.]

3. If $A^{mn} b_{mn} = 0$ show that there is an infinity of triads of tangent planes to the A cone which are mutually conjugate with respect to the B cone. Similarly for $a_{mn} B^{mn} = 0$.

[Choose a coordinate system such that B is reduced to canonical form and two of the coordinate planes touch A , and calculate the invariant in this system.]

4. If we are given two conics $\alpha^{mn} u_m u_n = 0$ and $\beta^{mn} u_m u_n$, examine the elementary divisors of $|\alpha^{mn} - \theta \beta^{mn}|$ in each of the cases of Ex. 3 (p. 97) and hence distinguish these cases analytically.

5. Show that the coefficients of the cubic polynomial $|\alpha^{mn} - \theta \beta^{mn}|$ are relative invariants of weight -2 , and find what their vanishing signifies geometrically.

EXAMPLES VII.

1. If a root of $|a_{mn} - \theta b_{mn}| = 0$ is of a certain multiplicity for all the first minors prove that it must be a multiple root of the determinant of a higher order.

2. Show that the necessary and sufficient condition that the cones A and B should have double contact is that (3), p. 89, should have a double root which is also a root of all the first minors.

3. Show that A and B have simple contact if (3), p. 89, has a double root which is not a root of all the first minors.

4. The cones A and B will have contact of the second order if (3), p. 89, has a triple root which is not a root of all the first minors.

5. If the cone B is the isotropic cone, prove the following properties of the family $(a_{mn} - \theta g_{mn}) x^m x^n = 0$:

(i) Show that the canonical form must come under section I (p. 92), that is, the form must be, (in rectangular Cartesians),

$$(\theta_1 - \theta) (\bar{x}^1)^2 + (\theta_2 - \theta) (\bar{x}^2)^2 + (\theta_3 - \theta) (\bar{x}^3)^2 = 0.$$

[This reduction has been already carried out at p. 83.]

(ii) The common polar directions are the principal axes of A .

(iii) The degenerate members are pairs of planes whose lines of intersection are the principal axes.

(iv) The cone A is of revolution if it has double contact with the isotropic cone.

6. If we consider the family of conics $(\alpha^{mn} - \theta g^{mn}) u_m u_n = 0$ obtained from the conics $\alpha^{mn} u_m u_n = 0$ and the circle at infinity, show that they have the following properties:

(i) The canonical form is

$$(\theta_1 - \theta) (\bar{u}_1)^2 + (\theta_2 - \theta) (\bar{u}_2)^2 + (\theta_3 - \theta) (\bar{u}_3)^2 = 0.$$

[This reduction has been already indicated in Ex. 3 (p. 85).]

(ii) The common polar lines are conjugate with respect to the circle at infinity.

(iii) The degenerate members are three pairs of points, a pair on each of the common polar lines.

7. **Confocal Cones.** Let us consider the cones obtained by joining the conics of Ex. 6 to the origin. This is a system of *confocal cones*; show that they have the following properties:

(i) The canonical form of the family is

$$(\bar{x}^1)^2 + \frac{\cdot}{\cdot} + \frac{\cdot}{\cdot} + (\bar{x}^3)^2 = 0.$$

(ii) The family has the same principal planes, namely, those obtained by joining O to the common polar lines of the family of conics.

(iii) The degenerate members are three pairs of lines through O , a pair in each of the principal planes. These lines are called *the focal lines of the family*.

8. If $B = A^{mn} b_{mn} = 0$ show that the planes $b_{mn} x^m x^n = 0$ are conjugate with respect to the cone $a_{mn} x^m x^n = 0$.

9. Infer from Ex. 8 that the condition that the planes $b_{mn} x^m x^n = 0$ should be orthogonal is $g^{mn} b_{mn} = 0$.

10. What are the theorems dual to Exs. 8 and 9?

11. If the tangent planes from the point P to the cone B are conjugate with respect to the cone A , then P lies on the cone

$$A^{mn} (b_{mn} b_{rs} - b_{mr} b_{ns}) x^r x^s = 0.$$

Also show that the equation of this cone can be written in the equivalent form

$$\varepsilon_{rmp} \varepsilon_{sqn} A^{mn} B^{pq} x^r x^s = 0.$$

12. Find the locus of a point P such that the tangent planes from it to the cone B are orthogonal.

[This is the particular case of Ex. 11, where $a_{mn} = g_{mn}$.]

13. Find the locus of P such that the pairs of tangent planes from it to the cones A and B form a harmonic range.

[This is the same problem as Ex. 11.]

14. If the polar planes of P with respect to the cones A and B are orthogonal show that P lies on the cone

$$g^{mn} a_{mr} b_{ns} x^r x^s = 0.$$

15. If the polar plane of P with respect to the cone B touches the cone A show that the locus of P is the cone

$$A^{mn} b_{mr} b_{ns} x^r x^s = 0.$$

CHAPTER VIII

CENTRAL QUADRICS

1. The point equation of a central quadric.

A *central quadric* is a surface whose equation is of the form

$$a_{mn} (x^m - x_0^m) (x^n - x_0^n) = 1. \quad \dots \quad (1)$$

The point x_0^r is called *the centre of the quadric*. This point may without loss of generality be taken as the origin, in which case the equation becomes

$$a_{mn} x^m x^n = 1. \quad \dots \quad (2)$$

We see immediately that if a_{mn} is taken to be symmetric, as it always can be, then it is a covariant double tensor.

If x_0^r is any point and a line be drawn through it in the direction λ^r , the equation of this line is

$$x^r = x_0^r + \rho \lambda^r,$$

where ρ is the distance between the points x_0^r and x^r , and the line cuts the quadric in the points given by the equation

$$\rho^2 a_{mn} \lambda^m \lambda^n + 2 \rho a_{mn} \lambda^m x_0^n + a_{mn} x_0^m x_0^n = 1. \quad \dots \quad (3)$$

The roots of this equation are the distances from x_0^r to the points of intersection of the line with the quadric, which we shall call Q for brevity.

From this equation we make the following observations:

(i) Defining the *polar plane of x'_0 with respect to Q* as the locus of all points x'' such that the points x'_0 , x'' form a harmonic range with the points of intersection, we find that the equation of the polar plane of x'_0 with respect to Q is

$$a_{mn} x''^m x''^n = 1. \quad (4)$$

(ii) If x'_0 lies on the quadric, the polar plane of x'_0 coincides with the *tangent plane to Q at x'_0* , and the equation of the tangent plane at x'_0 is (4), where x'_0 is now a point on the quadric.

(iii) If x'_0 is the centre of Q , then $x'_0 = 0$, and the equation (3) becomes

$$a_{mn} \lambda^m \lambda^n \varrho^2 = 1, \quad (5)$$

that is, all chords of Q through the centre are bisected at the centre.

(iv) The line through x'_0 will touch Q if

$$(a_{mn} \lambda^m x'_0{}^n)^2 = (a_{mn} \lambda^m \lambda^n) (a_{rs} x'_0{}^r x'_0{}^s - 1).$$

Hence the equation of the tangent cone from x'_0 to Q is

$$[a_{mn} (x''^m - x'_0{}^m) (x''^n - x'_0{}^n)]^2 = a_{mn} (x''^m - x'_0{}^m) (x''^n - x'_0{}^n) (a_{rs} x'_0{}^r x'_0{}^s - 1),$$

or, as it may be written,

$$(a_{mn} x''^m x''^n - 1)^2 = (a_{mn} x''^m x''^n - 1) (a_{rs} x'_0{}^r x'_0{}^s - 1). \quad (6)$$

EXAMPLES.

1. The distance from the centre to the quadric in the direction λ^r is called the *semi-diameter* in that direction. Show that this distance is $1/\sqrt{a_{mn} \lambda^m \lambda^n}$.

2. Show that $1/\sqrt{a_{11}}$, $1/\sqrt{a_{22}}$, $1/\sqrt{a_{33}}$ are the semi-diameters in the directions of the coordinate axes, provided the axes are Cartesian.

3. If λ^r is a generator of the cone $a_{mn} x''^m x''^n = 0$, show that both the roots of (5) are infinite. This cone is called the *asymptotic cone*.

2. The tangential equation of a central quadric.

Let us now find the condition that the plane

$$u_r x''^r = 1 \quad \dots \dots \dots (7)$$

should touch the quadric Q . If x'_0 is the point of contact the tangent plane to Q is

$$a_{mn} x''^m x''^n = 1,$$

and this plane is to be identical with (7). Hence

$$u_r = a_{r,n} x_0'''. \quad (8)$$

Multiplying across by A^{rs} , the cofactor of a_{rs} in A , we have

$$A^{rs} u_r = A x_0^s.$$

If $A \neq 0$ and if we put $\alpha^{rs} = A^{rs}/A$, we get

$$x_0^r = \alpha^{rs} u_s, \quad (9)$$

But

$$u_r x_0^r = 1.$$

Hence

$$\alpha^{mn} u_m u_n = 1, \quad (10)$$

which is the condition that the plane u_r must satisfy if it touches Q . We therefore call (10) the *plane* or *tangential equation of the quadric*.

Conversely, if the plane u_r satisfies the relation (10), we shall show that the plane touches a quadric surface. Let u_r^0 be a particular value satisfying (10), and let us consider the point

$$\alpha^{rs} u_r u_s^0 = 1. \quad (11)$$

If we denote this point by x_0^r , then

$$x_0^r = \alpha^{rs} u_s^0.$$

If $|\alpha^{rs}| \neq 0$, we deduce that

$$u_r^0 = a_{rs} x_0^s,$$

where a_{rs} is the cofactor of α^{rs} in $|\alpha^{rs}|$, divided by $|\alpha^{rs}|$. But u_r^0 satisfies (10). Hence

$$a_{rs} x_0^r x_0^s = 1, \quad (12)$$

and the plane u_r^0 has for its point equation

$$a_{rs} x^r x_0^s = 1.$$

In other words, the plane u_r^0 is a tangent plane to the surface

$$a_{mn} x^m x^n = 1$$

and the point of contact is

$$\alpha^{rs} u_r u_s^0 = 1.$$

So far we have considered only the cases where $|a_{mn}|$ and $|\alpha^{mn}|$ are not zero. The other cases we shall leave for the present.

EXAMPLES.

1. Show that the pole of the plane u_r^0 with respect to $\alpha^{mn} u_m u_n = 1$ is $\alpha^{rs} u_r u_s^0 = 1$.

2. Show that the centre is the pole of the plane at infinity.

3. Prove that the equation of the conic in which the plane u_r^0 cuts the quadric is given by

$$(\alpha^{mn} u_m u_n^0 - 1)^2 = (\alpha^{mn} u_m u_n - 1)(\alpha^{rs} u_r^0 u_s^0 - 1).$$

3. Canonical form of the equation of a quadric. Principal axes.

We have seen that the quadratic form $a_{mn} x^m x^n$ can be reduced in an infinity of ways to the form $\bar{a}_{11} (\bar{x}^1)^2 + \bar{a}_{22} (\bar{x}^2)^2 + \bar{a}_{33} (\bar{x}^3)^2$. In this case the equation of the quadric becomes

$$\bar{a}_{11} (\bar{x}^1)^2 + \bar{a}_{22} (\bar{x}^2)^2 + \bar{a}_{33} (\bar{x}^3)^2 = 1. \quad (13)$$

We call this form *the canonical form of the equation*. Moreover we know that we can always reduce the equation to this form and at the same time choose the axes to be rectangular Cartesian. In this last case the coordinate axes are called *the principal axes* and the coordinate planes are called *the principal planes of the quadric*. Also the quantities

$$\frac{1}{\sqrt{\bar{a}_{11}}}, \quad \frac{1}{\sqrt{\bar{a}_{22}}}, \quad \frac{1}{\sqrt{\bar{a}_{33}}},$$

which are the semi-diameters of the quadric in the directions of the principal axes are called the lengths of the *principal semi-axes*.

Similarly we have seen that the form $\alpha^{mn} u_m u_n$ can be reduced to the form $\bar{\alpha}^{11} (\bar{u}_1)^2 + \bar{\alpha}^{22} (\bar{u}_2)^2 + \bar{\alpha}^{33} (\bar{u}_3)^2$. Hence the plane equation of the second degree can be reduced to the canonical form

$$\bar{\alpha}^{11} (\bar{u}_1)^2 + \bar{\alpha}^{22} (\bar{u}_2)^2 + \bar{\alpha}^{33} (\bar{u}_3)^2 = 1 \quad (14)$$

in an infinity of ways. Moreover we can, in at least one way, reduce the equation to (14) and at the same time choose the axes to be rectangular Cartesian.

EXAMPLES.

1. Show that the principal semi-axes of the quadric (2) are $1/|\bar{\theta}_1|$, $1/|\bar{\theta}_2|$, $1/|\bar{\theta}_3|$, where θ_1 , θ_2 , θ_3 are the roots of $|a_{mn} - \theta b_{mn}| = 0$.

2. Show that the coordinates of the tangent plane to the quadric (13) satisfy the equation

$$\frac{(\bar{u}_1)^2}{\bar{a}_{11}} + \frac{(\bar{u}_2)^2}{\bar{a}_{22}} + \frac{(\bar{u}_3)^2}{\bar{a}_{33}} = 1.$$

3. Show that the plane, whose coordinates satisfy (14), touches the quadric

$$\frac{(\bar{x}^1)^2}{\bar{a}_{11}} + \frac{(\bar{x}^2)^2}{\bar{a}_{22}} + \frac{(\bar{x}^3)^2}{\bar{a}_{33}} = 1.$$

4. Two points are *conjugate* with respect to a quadric if one lies on the polar plane of the other. Show that the condition that x_r^1 and x_s^2 be conjugate with respect to $a_{mn} x^m x^n = 1$ is $a_{mn} x_1^m x_2^n = 1$.

5. Two planes are *conjugate* if the pole of one lies in the other. Show that u_r^1, u_s^2 are conjugate with respect to $\alpha^{mn} u_m u_n = 1$ if $\alpha^{mn} u_m^1 u_n^2 = 1$.

4. Classification of the central quadrics.

By taking the principal axes of the quadric as coordinate axes and thus reducing its equation to canonical form we can classify the different possible types of central quadrics. Let us take the determinant

$$A \equiv |a_{mn}| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

When the equation is in canonical form we have

$$\bar{A} \equiv \begin{vmatrix} \bar{a}_{11} & 0 & 0 \\ 0 & \bar{a}_{22} & 0 \\ 0 & 0 & \bar{a}_{33} \end{vmatrix} \equiv \bar{a}_{11} \bar{a}_{22} \bar{a}_{33} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

We divide the discussion into the following cases.

I. $A \neq 0$. It follows that \bar{A} is not zero, that is, none of the \bar{a} 's vanish, and the equation of the quadric is

$$\bar{a}_{11} (\bar{x}^1)^2 + \bar{a}_{22} (\bar{x}^2)^2 + \bar{a}_{33} (\bar{x}^3)^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

There are four subcases according to the signs of the coefficients:

- (i) If all three are positive, the quadric is an *ellipsoid*.
- (ii) If two are positive and one negative, the quadric is a *hyperboloid of one sheet*.
- (iii) If one is positive and two negative, the quadric is a *hyperboloid of two sheets*.
- (iv) If all three are negative, the quadric is *imaginary*.

II. $A = 0$, but all the first minors not zero. In this case $\bar{A} = 0$ but all the first minors of \bar{A} are not zero. Thus one only of the \bar{a} 's is zero, say \bar{a}_{33} . The equation of the quadric is

$$\bar{a}_{11} (\bar{x}^1)^2 + \bar{a}_{22} (\bar{x}^2)^2 = 1, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and the quadric is a *cylinder*, whose axis is the \bar{x}^3 -axis and whose cross-section is the conic given by (18) together with $\bar{x}^3 = 0$.

- (i) If $\bar{a}_{11}, \bar{a}_{22}$ are both positive, the cylinder is *elliptic*.
- (ii) If one is positive and one negative, the cylinder is *hyperbolic*.
- (iii) If both are negative, the cylinder is *imaginary*.

II. $A = 0$, $A^{mn} = 0$. Here $\bar{A} = 0$, $\bar{A}^{mn} = 0$, and therefore two of the \bar{a} 's are zero, say \bar{a}_{22} , \bar{a}_{33} . The equation of the quadric becomes

$$\bar{a}_{11} (\bar{x}^1)^2 = 1 \quad \dots \dots \dots (19)$$

and the quadric is a *pair of parallel planes*.

- (i) If \bar{a}_{11} is positive, the planes are *real*.
- (ii) If \bar{a}_{11} is negative, the planes are *imaginary*.

EXAMPLES.

1. To examine what the plane equation $\alpha^{11} (u_1)^2 + \alpha^{22} (u_2)^2 = 1$ represents.

Let u_0^0 be a plane whose coordinates satisfy the equation

$$\alpha^{11} (u_1)^2 + \alpha^{22} (u_2)^2 = 1, \quad \dots \dots \dots (\alpha)$$

and let us consider the point

$$\alpha^{11} u_1 u_1^0 + \alpha^{22} u_2 u_2^0 = 1.$$

If we denote its coordinates by x_0^r we have

$$x_0^3 = 0$$

or

$$u_1^0 = \frac{x_0^1}{\alpha^{11}}, \quad u_2^0 = \frac{x_0^2}{\alpha^{22}}, \quad x_0^3 = 0.$$

But u_1^0 , u_2^0 satisfies (α) , and hence we have

$$\frac{(x_0^1)^2}{\alpha^{11}} + \frac{(x_0^2)^2}{\alpha^{22}} - 1, \quad x_0^3 = 0,$$

that is, x_0^r lies on the conic $(x^1)^2/\alpha^{11} + (x^2)^2/\alpha^{22} = 1$, $x^3 = 0$. Also the plane u_0^0 has for its equation in point coordinates $u_0^r x^r = 1$, or

$$\frac{x^1 x_0^1}{\alpha^{11}} + \frac{x^2 x_0^2}{\alpha^{22}} + u_0^3 x^3 = 1,$$

and we see that this plane for all values of u_0^3 touches the conic at the point x_0^r . Hence (α) expresses the condition that the plane u_r touches a conic, or, in other words, (α) is the *plane equation of a conic*, which is in the plane $x^3 = 0$ and has the origin as centre.

2. Show that $\alpha^{11} (u_1)^2 = 1$ represents two points on the x^1 -axis having the origin as the middle point between them.

3. Classify the different types of surfaces represented by the equation $\alpha^{mn} u_m u_n = 1$, showing that we obtain the following cases:

- I. $|\alpha^{mn}| \neq 0$; a quadric whose centre is O .
- II. $|\alpha^{mn}| = 0$, but all the first minors not zero; a conic with centre at O .
- III. $|\alpha^{mn}| = 0$ and all the first minors zero; a pair of points, reflections of each other in the origin.

4. If $A = 0$, show that the axis of the cylinder $a_{mn} x^m x^n = 1$ is given by λ^r where $a_{rm} \lambda^m = 0$.

5. If all the first minors of $|a_{mn}|$ are zero, show that $a_{mn} x^m x^n$ is a perfect square.

5. Confocal quadrics.

Let

$$\alpha^{mn} u_m u_n = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

be the tangential equation of a central quadric and let us take the circle at infinity

$$g^{mn} u_m u_n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Every plane u_r which touches both (20) and (21) will also touch every member of the family

$$(\alpha^{mn} - \theta g^{mn}) u_m u_n = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

We call such a family a system of *confocal quadrics*, and we see that it is a family of concentric quadrics, O being the common centre.

The family has four degenerate members. One is the circle at infinity, which is given when the parameter θ is infinite, and the other three are the members given by the values of θ which satisfy the equation

$$|\alpha^{mn} - \theta g^{mn}| = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

These members we know are conics with their centres at O , and they are called the *focal conics of the system*.

We shall now obtain the canonical form of the equation of a confocal system. We can always reduce the two quadratic forms $\alpha^{mn} u_m u_n$ and $g^{mn} u_m u_n$, of which the second is positive definite, simultaneously to the forms

$$\bar{\alpha}^{11} (\bar{u}_1)^2 + \bar{\alpha}^{22} (\bar{u}_2)^2 + \bar{\alpha}^{33} (\bar{u}_3)^2 \quad \text{and} \quad (\bar{u}_1)^2 + (\bar{u}_2)^2 + (\bar{u}_3)^2.$$

Hence the equation of the family becomes

$$(\bar{\alpha}^{11} - \theta) (\bar{u}_1)^2 + (\bar{\alpha}^{22} - \theta) (\bar{u}_2)^2 + (\bar{\alpha}^{33} - \theta) (\bar{u}_3)^2 = 1.$$

Also the values $\bar{\alpha}^{11}$, $\bar{\alpha}^{22}$, $\bar{\alpha}^{33}$ are the roots of (23), so that if we denote these roots by θ_1 , θ_2 , θ_3 the equation of the family is

$$(\theta_1 - \theta) (\bar{u}_1)^2 + (\theta_2 - \theta) (\bar{u}_2)^2 + (\theta_3 - \theta) (\bar{u}_3)^2 = 1. \quad (24)$$

Moreover, since g^{mn} is the metric tensor we see that the new coordinate system is rectangular and Cartesian. The corresponding point equation of (24) is

$$\frac{(\bar{x}^1)^2}{\theta_1 - \theta} + \frac{(\bar{x}^2)^2}{\theta_2 - \theta} + \frac{(\bar{x}^3)^2}{\theta_3 - \theta} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and it is seen that all the members have the same principal axes and planes. The focal conics are obtained by putting $\theta = \theta_1, \theta_2, \theta_3$ in (24). For example, putting $\theta = \theta_3$, we have the conic

$$(\theta_1 - \theta_3)(\bar{u}_1)^2 + (\theta_2 - \theta_3)(\bar{u}_2)^2 = 1,$$

and we know that the corresponding point equations are

$$\frac{\bar{x}^2}{\theta_1 - \theta_3} + \frac{\bar{y}^2}{\theta_2 - \theta_3} = 1, \quad \bar{x}^3 = 0,$$

which is a conic in the principal plane $\bar{x}^3 = 0$. Similarly the other two conics are in the remaining principal planes. *Thus the focal conics lie on the common principal planes of the family.*

EXAMPLES.

1. Show that only one confocal can be drawn to touch a given plane.
2. Show that the locus of the pole of a given plane with respect to a system of confocals is a straight line.
3. Show that the tangent planes to two confocals at any point of their intersection are at right angles.

EXAMPLES VIII.

1. Show that the line $x^r = x_0^r + \rho \lambda^r$ lies wholly on the quadric if

$$a_{mn} x_0^m x_0^n = 1, \quad a_{mn} x_0^m \lambda^n = 0, \quad a_{mn} \lambda^m \lambda^n = 0.$$

Such a line is called a *generator* and we see that the generators of a quadric are parallel to the generators of the asymptotic cone.

2. Two diameters or lines through the centre are *conjugate* when their directions satisfy the relation $a_{mn} \lambda_1^m \lambda_2^n = 0$. Show that the locus of all diameters conjugate to λ^r is the plane $a_{mn} \lambda^m x^n = 0$, which is called the *plane conjugate to λ^r* .

3. If the axes are conjugate with respect to the quadric, then $a_{12} = a_{23} = a_{31} = 0$. Hence when the quadric is expressed in canonical form the coordinate axes are conjugate with respect to the quadric.

4. By examining the invariant $g_{mn} \alpha^m \alpha^n$ in coordinates which are Cartesian and in which the axes are conjugate with respect to the quadric, show that the sum of the squares of three conjugate semi-diameters is constant.

$$\left[\text{The invariant in such a system is } \frac{1}{a_{11}} + \frac{1}{a_{22}} + \frac{1}{a_{33}} \right].$$

5. By examining the invariant $g|\alpha^m \alpha^n|$ in the same coordinate system show that the tetrahedron, whose vertices are the extremities of three conjugate semi-diameters and the origin, has a constant volume.

6. Examine the invariant $g^{mn}a_{mn}$ in coordinates which are rectangular Cartesian and deduce that the sum of the squares of the reciprocals of three semi-diameters, which are mutually orthogonal, is constant.

7. Show that the tangential equation of the unit sphere $g_{mn}x^m x^n = 1$ is $g^{mn}u_m u_n = 1$ and show that the circle in which it is cut by the plane at infinity is the circle at infinity.

8. Show that the principal axes of a quadric are given by

$$a_{rm}\lambda^m = \theta\lambda_r$$

where θ is a root of $|a_{mn} - \theta g_{mn}| = 0$.

[This is a vector equation, and by considering the case where the coordinate axes are the principal axes (p. 107) we see that it is satisfied only by the principal axes.]

9. Show that the direction perpendicular to the polar plane of x_0^r with respect to the quadric is given by the vector $a_{rs}^r x_0^s$.

10. Prove that the conditions that $a_{mn}x^m x^n = 1$ should be of revolution is that there exists a value of θ which makes all the first minors of $|a_{mn} - \theta g_{mn}|$ vanish. Express these conditions explicitly when the coordinates are rectangular and Cartesian.

11. If $a_{mn}x^m x^n = 1$ is a right circular cylinder show that the conditions of Ex. 10 are satisfied together with $A = 0$.

12. Examine the surface

$$(a_{mn}a_{rs} - a_{mr}a_{ns})\lambda^m\lambda^n x^r x^s = a_{mn}\lambda^m\lambda^n,$$

when the coordinates are chosen such that λ^r is the \bar{x}^3 -axis and at the same time the quadric $a_{mn}x^m x^n = 1$ is reduced to canonical form. Hence show that it is the equation of the cylinder which envelopes the quadric and whose axis is in the direction λ^r .

13. To investigate *plane sections of a central quadric*, let λ^r be the direction conjugate to the given plane and let us take our axes such that λ^r becomes the \bar{x}^3 -axis and the quadric reduces to its canonical form. Prove that the given plane becomes parallel to $\bar{x}^3 = 0$ and that all parallel sections are similar.

14. If $\xi_r x^r = 1$ is a plane cutting the quadric, show that the cylinder passing through this section with its axis in the direction conjugate to the plane is

$$\alpha^{rs}\xi_r\xi_s\{a_{mn}x^m x^n - 1\} = (\xi_r x^r)^2 - 1.$$

15. Show that the principal semi-axes of the section of the quadric by $\xi_r x^r = 0$ are $1/|\bar{\theta}_1|$, $1/|\bar{\theta}_2|$, where θ_1, θ_2 are the roots of

$$\begin{vmatrix} a_{mn} - \theta g_{mn} & \xi_m \\ \xi_n & 0 \end{vmatrix} = 0,$$

and that the principal directions λ^r of the section are given by

$$\begin{aligned} (a_{rm} - \theta g_{rm})\lambda^m &= k\xi_r, \\ \xi_m\lambda^m &= 0. \end{aligned}$$

[These are vector equations. Examine them in the coordinates suggested in Ex. 13.]

16. Circular Sections. If we choose θ so that it satisfies the equation $|a_{mn} - \theta g_{mn}| = 0$ show that the planes $(a_{mn} - \theta g_{mn}) x^m x^n = 0$ cut the quadric $a_{mn} x^m x^n = 1$ in circular sections, and show that these planes intersect in pairs in the principal axes of the quadric.

[Reduce the quadric to its canonical form, the axes being the principal axes.]

CHAPTER IX

THE GENERAL QUADRIC

1. The general equation of a quadric.

The general equation of a quadric is

$$a_{mn} x^m x^n + 2 b_m x^m + c = 0 \quad (1)$$

where a_{mn} is a double symmetric tensor, b_m is a vector and c an invariant, all having constant values.

If x_0^r is any point P and we draw a straight line through P in the direction λ^r cutting the quadric in points R, S , then the lengths of PR and PS are the roots of the equation

$$\begin{aligned} \varrho^2 a_{mn} \lambda^m \lambda^n + 2 \varrho (a_{mn} x_0^m \lambda^n + b_m \lambda^m) + a_{mn} x_0^m x_0^n \\ + 2 b_m x_0^m + c = 0. \end{aligned} \quad (2)$$

From this equation we have the following results.

(i) The locus of a point Q such that $(PRQS)$ forms a harmonic range is the plane

$$\begin{aligned} a_{mn} x_0^m (x^n - x_0^n) + b_m (x^m - x_0^m) \\ = -[a_{mn} x_0^m x_0^n + 2 b_m x_0^m + c], \end{aligned}$$

or

$$a_{mn} x^m x_0^n + b_m (x^m + x_0^m) + c = 0. \quad (3)$$

This is the *polar plane of P with respect to the quadric*. We define two points P, Q as being *conjugate* with respect to the quadric when the polar plane of one passes through the other, and we see that the condition that x_1^r and x_2^r be conjugate with respect to (1) is that

$$a_{mn} x_1^m x_2^n + b_m (x_1^m + x_2^m) + c = 0 \quad (4)$$

(ii) If P lies on the quadric, one root of (2) is already zero and a second is zero if Q (any point on PRS) lies on the plane

$$a_{mn} x_0^m (x^n - x_0^n) + b_m (x^m - x_0^m) = 0,$$

or, as it may be written,

$$a_{mn} x^m x_0^n + b_m (x^m + x_0^m) + c = 0. \quad \dots \quad (5)$$

This is therefore *the equation of the tangent plane* at P , and we see that it is also the polar plane of P .

(iii) If P is not on the quadric, the equation (2) has equal roots if Q lies on the tangent cone from P to the quadric. That the roots may be equal Q must lie on the cone

$$\begin{aligned} [a_{mn} x_0^m (x^n - x_0^n) + b_m (x^m - x_0^m)]^2 \\ = a_{mn} (x^m - x_0^m) (x^n - x_0^n) (a_{mn} x_0^m x_0^n + 2 b_m x_0^m + c), \end{aligned}$$

that is, on the cone

$$\begin{aligned} [a_{mn} x^m x_0^n + b_m (x^m + x_0^m) + c]^2 \\ = (a_{mn} x^m x^n + 2 b_m x^m + c) (a_{mn} x_0^m x_0^n + 2 b_m x_0^m + c) \quad (6) \end{aligned}$$

which we therefore see is the equation of *the tangent cone* from P to the quadric.

EXAMPLES.

1. If we draw a line through P parallel to a generator of the cone

$$a_{mn} x^m x^n = 0,$$

one of the intersections with (1) is at infinity.

2. The line $x^r = x_0^r + \rho \lambda^r$ will lie wholly on the surface if

$$\begin{aligned} a_{mn} x_0^m x_0^n + 2 b_m x_0^m + c = 0, \quad a_{mn} \lambda^m x_0^n + b_m \lambda^m = 0, \\ a_{mn} \lambda^m \lambda^n = 0. \end{aligned}$$

Such a line is called a *generator of the quadric*.

3. Show that the middle points of all chords drawn in a fixed direction λ^r lie on the plane $a_{mn} x^m \lambda^n + b_m \lambda^m = 0$.

4. Show that all chords which are bisected at a given point x_0^r lie on the plane $(a_{mn} x_0^m + b_n) (x^n - x_0^n) = 0$.

2. The centre.

If we can choose the point x_0^r such that

$$a_{rm} x_0^m + b_r = 0, \quad \dots \quad (7)$$

we see from (2) that every line through this point cuts the quadric in a segment which is bisected at P . Such a point P is called a *centre of the quadric*. Hence the existence of a centre depends on the existence of a solution of (7).

These are the equations of three planes and we have discussed the relations of three planes at p. 58. From the discussion given there we infer the following results about the existence of a centre for the quadric (1):

I. $A \equiv |a_{mn}| \neq 0$. There is a unique finite point of intersection of the planes and consequently there is a *unique finite centre*.

II. $A = 0$, but all the first minors of A are not zero. There are two subcases:

(i) $b_m A^{mr} \neq 0$, where A^{mn} is the cofactor of a_{mn} in A . There is a *unique centre at infinity*.

(ii) $b_m A^{mr} = 0$. There is a *finite line of centres*.

III. $A = 0$, and all the first minors are zero. Again there are two subcases:

(i) $a_{rs} \neq k b_r b_s$. There is a *line of centres at infinity*.

(ii) $a_{rs} = k b_r b_s$. There is a *plane of centres*.

EXAMPLES.

1. If $A \neq 0$, show that the unique centre is given by $x_0^r = -\alpha^{rs} b_s$, where α^{rs} is the cofactor of a_{rs} in A , divided by A .

2. Show that

$$A(c - \alpha^{mn} b_m b_n) = \begin{vmatrix} a_{mn} & b_m \\ b_n & c \end{vmatrix} = \Delta.$$

3. If $A \neq 0$, show that the equation of the quadric can be written

$$a_{mn}(x^m - x_0^m)(x^n - x_0^n) + \frac{\Delta}{A} = 0,$$

where x_0^r is the centre. This is the equation of a central quadric.

4. In Ex. 3 show that if $\Delta = 0$ then the quadric is a cone with vertex at x_0^r .

3. The reduction of the equation of a quadric.

We shall now turn to the problem of reducing the general equation (1) to the different canonical forms in which it can be expressed. We shall use the notation

$$\Delta = \begin{vmatrix} a_{mn} & b_m \\ b_n & c \end{vmatrix}, \quad \dots \dots \dots (8)$$

and as before Δ will denote $|a_{mn}|$ and A^{mn} its cofactors.

We have seen that the quadratic form $a_{mn} x^m x^n$ can be reduced to the form

$$\bar{a}_{11}(\bar{x}^1)^2 + \bar{a}_{22}(\bar{x}^2)^2 + \bar{a}_{33}(\bar{x}^3)^2$$

in an infinity of ways, and for at least one of these the axes are rectangular Cartesian. We shall suppose one of these transformations has been effected, and we divide the discussion into the following cases.

I. $A \neq 0$. The quantities \bar{a}_{11} , \bar{a}_{22} , \bar{a}_{33} are not zero, and the quadric can be written

$$\bar{a}_{11} \left(\bar{x}^1 + \frac{\bar{b}_1}{\bar{a}_{11}} \right)^2 + \bar{a}_{22} \left(\bar{x}^2 + \frac{\bar{b}_2}{\bar{a}_{22}} \right)^2 + \bar{a}_{33} \left(\bar{x}^3 + \frac{\bar{b}_3}{\bar{a}_{33}} \right)^2 + c - \sum \frac{(\bar{b}_i)^2}{\bar{a}_{ii}} = 0.$$

Also

$$\bar{\Delta} = \bar{a}_{11} \bar{a}_{22} \bar{a}_{33} \left(c - \sum \frac{(\bar{b}_i)^2}{\bar{a}_{ii}} \right)$$

Hence we can write the equation in the form

$$\bar{a}_{11} (\bar{x}^1 - \bar{x}_0^1)^2 + \bar{a}_{22} (\bar{x}^2 - \bar{x}_0^2)^2 + \bar{a}_{33} (\bar{x}^3 - \bar{x}_0^3)^2 + \frac{\bar{\Delta}}{\bar{A}} = 0. \quad (9)$$

(i) $\bar{\Delta} \neq 0$. Then $\bar{\Delta} \neq 0$ and the equation (9) is the equation of a *central quadric* whose centre is the point \bar{x}_0^r .

(ii) $\bar{\Delta} = 0$. Then $\bar{\Delta} = 0$ and (9) is the equation of a *proper cone* whose vertex is the point \bar{x}_0^r .

II. $A = 0$, but all its first minors do not vanish. Here one and only one of the \bar{a} 's is zero. Let it be \bar{a}_{33} which vanishes. The equation of the quadric becomes

$$\bar{a}_{11} \left(\bar{x}^1 + \frac{\bar{b}_1}{\bar{a}_{11}} \right)^2 + \bar{a}_{22} \left(\bar{x}^2 + \frac{\bar{b}_2}{\bar{a}_{22}} \right)^2 + 2 \bar{b}_3 \bar{x}^3 + c - \frac{\bar{b}_1^2}{\bar{a}_{11}} - \frac{\bar{b}_2^2}{\bar{a}_{22}} = 0.$$

Also

$$\bar{A}^{11} = \bar{A}^{22} = 0, \quad \bar{A}^{33} = \bar{a}_{11} \bar{a}_{22}, \quad \bar{A}^{mn} = 0 \quad (m \neq n).$$

(i) $\bar{b}_m \bar{A}^{mr} = 0$ are not all zero. Then $\bar{b}_m \bar{A}^{mr}$ are not all zero and hence $\bar{b}_3 \neq 0$. The surface can be written in the form

$$\bar{a}_{11} (\bar{x}^1 - \bar{x}_0^1)^2 + \bar{a}_{22} (\bar{x}^2 - \bar{x}_0^2)^2 + 2 \bar{b}_3 (\bar{x}^3 - \bar{x}_0^3) = 0. \quad (10)$$

The quadric is a *paraboloid*. Moreover if the axes are rectangular Cartesian the point \bar{x}_0^r is called the *vertex* and the parallel through it to the \bar{x}^3 -axis the *axis of the paraboloid*. From the previous section we see that there is a unique centre at infinity.

(ii) $\bar{b}_m \bar{A}^{mr} = 0$. Then $\bar{b}_m \bar{A}^{mr} = 0$ and consequently $\bar{b}_3 = 0$. The quadric becomes

$$\bar{a}_{11} (\bar{x}^1 - \bar{x}_0^1)^2 + \bar{a}_{22} (\bar{x}^2 - \bar{x}_0^2)^2 + c - \frac{\bar{b}_1^2}{\bar{a}_{11}} - \frac{\bar{b}_2^2}{\bar{a}_{22}} = 0. \quad (11)$$

This is the equation of a *cylinder* whose axis is the line through $(\bar{x}_0^1, \bar{x}_0^2, 0)$ parallel to the \bar{x}^3 -axis. There is a line of centres, namely, the axis of the cylinder. If it should happen that

$$c - \frac{\bar{b}_1^2}{\bar{a}_{11}} - \frac{\bar{b}_2^2}{\bar{a}_{22}} = 0,$$

the cylinder degenerates into *two intersecting planes*, the axis of the cylinder becoming the line of intersection. In this last particular case it is easy to verify that all the first minors of \bar{A} , and therefore of A also, are zero.

III. $A = 0$, and all its first minors vanish. Here two of the quantities \bar{a}_{11} , \bar{a}_{22} , \bar{a}_{33} vanish, say $\bar{a}_{11} = \bar{a}_{22} = 0$. The equation of the quadric reduces to

$$\bar{a}_{33} \left(\bar{x}^3 + \frac{\bar{b}_3}{\bar{a}_{33}} \right)^2 + 2 \bar{b}_1 \bar{x}^1 + 2 \bar{b}_2 \bar{x}^2 + c - \frac{\bar{b}_3^2}{\bar{a}_{33}} = 0,$$

Our choice of coordinates only restricts the \bar{x}^3 -plane and we can transform the \bar{x}^1 and \bar{x}^2 axes in this plane so that one of the quantities \bar{b}_1 , \bar{b}_2 becomes zero, say \bar{b}_1 .

(i) $\bar{a}_{mn} \neq k \bar{b}_m \bar{b}_n$. Then $\bar{a}_{mn} \neq k \bar{b}_m \bar{b}_n$, from which we conclude that $\bar{b}_2 \neq 0$. The equation can now be written

$$\bar{a}_{33} (\bar{x}^3 - \bar{x}_0^3)^2 + 2 \bar{b}_2 (\bar{x}^2 - \bar{x}_0^2) = 0. \quad (12)$$

The quadric is a *parabolic cylinder* whose axis is parallel to the \bar{x}^1 -axis. If the axes are rectangular Cartesian we call the line through $(0, \bar{x}_0^2, \bar{x}_0^3)$ parallel to the \bar{x}^1 -axis the *line of vertices*. There is a line of centres at infinity.

(ii) $\bar{a}_{mn} = k \bar{b}_m \bar{b}_n$. We now have $\bar{b}_2 = 0$, and the equation of the quadric is

$$\bar{a}_{33} (\bar{x}^3 - \bar{x}_0^3)^2 + \left(c - \frac{\bar{b}_3^2}{\bar{a}_{33}} \right) = 0. \quad (13)$$

The quadric becomes a *pair of parallel planes*. If it should happen that

$$\frac{\bar{b}_3^2}{\bar{a}_{33}} = 0,$$

then (13) is the equation of *two coincident planes*. There is a plane of centres.

This exhausts all the possible types of quadrics in point-coordinates.

EXAMPLES.

1. Show that the types of quadrics may be classified as follows:

I. $A \neq 0$. (i) $A \neq 0$, a central quadric; (ii) $A = 0$, $A^{mn} \neq 0$, a paraboloid.

II. $A = 0$, but all first minors do not vanish. (i) $A \neq 0$, a proper cone; (ii) $A = 0$, $A^{mn} \neq 0$, a cylinder (elliptic or hyperbolic); (iii) $A = A^{mn} = 0$, a parabolic cylinder.

III. $A = 0$, all first minors vanish but not all second minors. (i) $A = 0$, $A^{mn} \neq 0$, a pair of intersecting planes; (ii) $A = A^{mn} = 0$, a pair of parallel planes.

IV. $A = 0$ and all its second minors. A pair of coincident planes.

2. Discriminate between the cases according to the nature of the centre or centres.

3. Show that A and its cofactors are relative tensors of weight 2.

EXAMPLES IX.

1. Show that if x_0^r is a point on the general quadric (1), p. 113, then

$$a_{mn} x_0^m x_0^n + 2b_m x_0^m + c = a_{mn} (x_0^m - x_0^n)(x_0^n - x_0^m) \\ + 2[a_{mn} x_0^m x_0^n + b_m (x_0^m + x_0^n) + c].$$

Deduce that the tangent plane at x_0^r cuts the quadric in the same two lines as those in which it cuts the cone $a_{mn} (x_0^m - x_0^n)(x_0^n - x_0^m) = 0$.

2. Deduce from Ex. 1 that the tangent plane to a quadric cuts it in two generators which are parallel to two generators of the cone $a_{mn} x^m x^n = 0$ and which intersect in the point of contact.

3. Show that the plane coordinates of the polar plane of x_0^r with respect to the quadric are

$$-\frac{a_{rm} x_0^m + b_r}{c + b_m x_0^m}.$$

4. The polar planes of two points x_0^r, x_1^r with respect to the quadric intersect in a line called the *polar line* of the line joining the two points. Deduce from Ex. 3 that the six coordinates of the polar line of (λ^r, μ_r) are proportional to

$$\varepsilon^{rmn} a_{mp} b_n \lambda^p + \frac{1}{2} \varepsilon^{rmn} \varepsilon^{pq} a_{mp} a_{nq} \mu_t, \\ b_r b_m \lambda^m - \varepsilon^{pmn} \mu_p a_{rm} b_n - c a_{rm} \lambda^m.$$

5. Show that the normal vector to the quadric at the point x_0^r on it is proportional to

$$a_{rm} x_0^m + b_r.$$

6. Show that the equation of the enveloping cylinder whose axis is in the direction λ^r is

$$(a_{mn} x^m \lambda^n + b_m \lambda^m)^2 = a_{mn} \lambda^m \lambda^n (a_{mn} x^m x^n + 2b_m x^m + c).$$

7. If $a_{mn} = g_{mn}$, show that the quadric can always be reduced to the form (9), p. 116. It is the general equation of a *sphere*.

8. Show that the locus of the middle points of all chords drawn in a given direction λ^r is a plane through the centre.

9. If the reduction in § 3, p. 115, is to rectangular Cartesian coordinates we may say for brevity that the quadric has been reduced to its *principal canonical form*. Show that $\bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}$ in the principal canonical form are the roots of $|a_{mn} - \theta g_{mn}| = 0$.

10. If two of these \bar{a} 's in the principal form are equal the *quadric is of revolution*. Show that in this case $|a_{mn} - \theta g_{mn}| = 0$ has a double root which is also a root of all the first minors.

11. If $\Delta \neq 0, \Delta \neq 0$, show that the principal semi-axes of the central quadric are $|\Delta/\Delta \theta_1|^{1/2}, |\Delta/\Delta \theta_2|^{1/2}, |\Delta/\Delta \theta_3|^{1/2}$, $\theta_1, \theta_2, \theta_3$ being the roots of $|a_{mn} - \theta g_{mn}| = 0$.

12. If the quadric is a paraboloid, show that the direction λ^r of the axis is given by the equation $a_{rs} \lambda^s = 0$, and if λ^r is a unit vector show that the vertex of the paraboloid satisfies the equations

$$a_{rs} x^s + b_r = (b_m \lambda^m) \lambda_r, \quad (b_r + b_m \lambda^m \lambda_r) x^r + c = 0.$$

[These are tensor equations and, referring to (10), p. 116, we see that they are true when the quadric is in its principal canonical form.]

13. If the quadric is a cylinder or a pair of intersecting planes show that the axis is given by the intersection of the planes

$$a_{rm} x^m + b_r = 0$$

which have in this case a line in common.

14. If $ca_{mn} = b_m b_n$ show that the quadric reduces to two coincident planes and that the equation of this plane is $b_m x^m + c = 0$.

15. Show that the quantities

$$g^{mn} a_{mn}, \quad \frac{g_{mn} A^{mn}}{g}, \quad \frac{A}{g}, \quad \frac{\Delta}{g}$$

are all absolute invariants. Hence deduce that if we transform from one set of orthogonal Cartesian coordinates to another, then

$$a_{11} + a_{22} + a_{33}, \quad A^{11} + A^{22} + A^{33}, \quad A, \quad \Delta$$

are unchanged in value.

16. The condition that the plane $u_r x^r = 1$ touches the general quadric is

$$\begin{array}{ccc} a_{mn} & b_m & u_m \\ b_n & c & -1 \\ u_n & -1 & 0 \end{array} = 0.$$

17. Classify the different types of surfaces included in the plane equation $\alpha^{mn} u_m u_n + 2\beta^m u_m + \gamma = 0$. If we put

$$\Delta = \begin{vmatrix} \alpha^{mn} & \beta^m & \gamma \\ \beta^n & c & -1 \\ \gamma & -1 & 0 \end{vmatrix}, \quad \Delta = |\alpha^{mn}|,$$

show that we have the following classification.

I. $\Delta \neq 0$, (i) $\Delta \neq 0$, a quadric not passing through the origin; (ii) $\Delta = 0$, $\Delta^{mn} \neq 0$, a quadric passing through the origin.

II. $\Delta = 0$, but all the first minors do not vanish. (i) $\Delta \neq 0$, a conic whose plane does not contain the origin; (ii) $\Delta = 0$, $\Delta^{mn} \neq 0$, a conic in a plane through O . (iii) $\Delta = \Delta^{mn} = 0$, a conic passing through O .

III. $\Delta = 0$, all first minors vanish but not all second minors. (i) $\Delta = 0$, $\Delta^{mn} \neq 0$, a pair of points whose join does not pass through O ; (ii) $\Delta = \Delta^{mn} = 0$, a pair of points on a line through O .

IV. $\Delta = 0$ and all second minors vanish. A pair of coincident points.

18. Show that $\alpha^{mn} u_m u_n + 2\beta^m u_m = 0$ is the tangential equation of a paraboloid if $\Delta \neq 0$ and of a parabola if $\Delta = 0$.

CHAPTER X

AFFINE TRANSFORMATIONS

1. Affine transformations.

We have seen that a transformation of the form

$$X^r = a^r_s x^s + b^r, \quad (1)$$

where the a 's and b 's are constants, can be regarded as a change of rectilinear coordinate axes, that is, the quantities x^r , X^r are the coordinates of the *same* point referred to *different* systems of coordinates. These equations can be looked at from another point of view. We shall now regard them as transforming a point p , whose coordinates are x^r , into a point P , whose coordinates are X^r referred to the *same* coordinate system. Such a transformation is called an *affine transformation* and we see that points are transformed into points, planes into planes and straight lines into straight lines. But it is not the most general of such transformations. In fact, we see that the plane at infinity is transformed into itself. The transformation (1) is also often called a *homogeneous deformation* or *strain*, especially in the theory of elasticity.

We shall consider the transformation as taking place in two stages. First let us take the transformation

$$X^r_1 = a^r_s x^s, \quad (2)$$

which differs from (1) only in the absence of the b 's, and secondly we transform X^r_1 into X^r by the relations

$$X^r = X^r_1 + b^r. \quad (3)$$

These two transformations combined give (1). Now the second of these, namely (3), gives merely a *constant displacement*, equal to b^r , to every point of space and such a transformation we shall neglect, so that we consider for the present only transformations (1) in which the b 's are zero.

Since $a^r_s x^s$ is a contravariant vector for all values of x^r , we conclude that a^r_s is a mixed double tensor. Also we see that to every point p there is a unique point P . If the determinant

$$|a^r_s|, \quad (4)$$

which is an invariant, is not zero, we can solve the equations (1) uniquely for the x 's and the transformation is *reversible*, that is, every point P is the transform of a unique point p .

We notice that when the b 's are zero, the transformation leaves the origin unaltered.

EXAMPLES.

1. If $|a^r_s| \neq 0$, show that the inverse transformation is

$$x^r = \alpha^{r,s}_s (X^s - b^s),$$

where $\alpha^{r,s}_s$ is the cofactor of a^s_r in $|a^{nn}_n|$, divided by $|a^{nn}_n|$.

2. Show that two successive affine transformations are equivalent to an affine transformation and that the determinant of the combined transformation equals the product of the determinants of the two single transformations.

3. If $|a^r_s| = 0$, show that the transformation is not reversible.

[We can find values x^s_0 different from zero such that $a^r_s x^s_0 = 0$. Hence both the points x^r_0 and the origin are transformed into the same point b^r , which shows that the correspondence is not one-to-one.]

2. The quadric of a transformation.

We confine ourselves to the transformation

$$X^r = a^r_s x^s, \quad \dots \dots \dots (5)$$

in which the origin is unchanged. We shall suppose that g_{mn} is the fundamental metric tensor in our coordinate system, so that

$$\delta^2 = g_{mn} (x^m_1 - x^m_2) (x^n_1 - x^n_2)$$

gives the square of the distance between the points x^r_1 and x^r_2 .

Let us consider the locus of the points which after transformation lie on the unit sphere

$$g_{mn} X^m X^n = 1. \quad \dots \dots \dots (6)$$

The original points must satisfy the relation

$$g_{mn} a^m_r a^n_s x^r x^s = 1. \quad \dots \dots \dots (7)$$

This is the equation of a central quadric which we shall refer to as the quadric Q . Its principal axes are given by

$$(g_{mn} a^m_r a^n_s - \theta g_{rs}) \lambda^s = 0, \quad \dots \dots \dots (8)$$

where θ is a root of the equation

$$|g_{mn} a^m_r a^n_s - \theta g_{rs}| = 0. \quad \dots \dots \dots (9)$$

Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of this equation and $\lambda_{(1)}^r, \lambda_{(2)}^r, \lambda_{(3)}^r$ the corresponding principal directions. We know from the theory of quadrics (cf. p. 84) that

$$g_{mn} a_{.r}^m a_{.s}^n \lambda_{(i)}^r \lambda_{(j)}^s = \begin{cases} 0, & \text{if } i \neq j \\ \alpha_i & \text{if } i = j \end{cases} \quad (10)$$

Therefore we see that the vectors

$$a_{.s}^r \lambda_{(i)}^s,$$

which are the transformed vectors of the original triad $\lambda_{(i)}^r$, are mutually orthogonal and are of magnitudes $\sqrt{\alpha_i}$. Let these directions be denoted by the unit vectors $A_{(i)}^r$. We conclude that if

$$x^r = \sum_{i=1}^3 \xi^{(i)} \lambda_{(i)}^r \quad \dots \quad (11)$$

then

$$X^r = \sum_{i=1}^3 \sqrt{\alpha_i} \xi^{(i)} A_{(i)}^r. \quad \dots \quad (12)$$

Let us refer the points x^r to the directions $\lambda_{(i)}^r$ as axes, and the points X^r to $A_{(i)}^r$ as other axes. The connection between the coordinates of corresponding points is

$$X'^1 = \sqrt{\alpha_1} x^1, \quad X'^2 = \sqrt{\alpha_2} x^2, \quad X'^3 = \sqrt{\alpha_3} x^3, \quad \dots \quad (13)$$

where dashed letters refer to one set of axes and undashed to the other set, both systems of axes being rectangular Cartesian.

EXAMPLES.

1. Show that there always exists an orthogonal triad of directions which remain orthogonal after the transformation.
2. If $|a_{.s}^r| \neq 0$, show that the points of the unit sphere $g_{mn} x^m x^n = 1$ are transformed into points on the quadric

$$g_{mn} \alpha_r^m \alpha_s^n X^r X^s = 1.$$

Prove that $A_{(i)}^r$ are the principal directions of this quadric and the lengths of its semi-axes are $\sqrt{\alpha_i}$.

3. Show that the lengths of the semi-axes of Q are $1/\sqrt{\alpha_i}$.
4. Show that $g_{mn} a_{.r}^m a_{.s}^n$ and $g_{mn} \alpha_r^m \alpha_s^n$ are symmetric tensors.
5. If Op cuts the quadric Q in the point q , show that the length of Op is stretched by the transformation in the ratio $1/Oq$.

3. Pure Strain.

Let us now consider transformations in which the orthogonal triad of directions $\lambda_{(i)}^r$, which remain orthogonal after transformation, are also invariable in direction. In this particular case

$$A_{(i)}^r = \lambda_{(i)}^r, \quad \dots \dots \dots (14)$$

and

$$a_{r,s} \lambda_{(i)}^s = \sqrt{\alpha_i} \lambda_{(i)}^r, \quad [\text{not summed for } i] \quad \dots \dots \dots (15)$$

Let us choose axes which are Cartesian and rectangular and which coincide with $\lambda_{(i)}^r$. We see that in this coordinate system

$$a_{r,s} = \sqrt{\alpha_s} \delta_s^r, \quad [\text{not summed for } s] \dots \dots \dots (16)$$

Consequently the double covariant tensor, $a_{r,s} = g_{r,p} a^p_s$, has for its components in this system

$$a_{r,s} = \sqrt{\alpha_s} g_{r,s}, \quad [\text{not summed for } s], \quad \dots \dots \dots (17)$$

from which we conclude that $a_{r,s}$ is a symmetric tensor. If we refer to the quadric

$$a_{mn} x^m x^n = 1 \quad \dots \dots \dots (18)$$

as the quadric Q_1 , we see from (15) that the axes of these quadrics Q and Q_1 coincide and that the semi-axes of Q_1 are

$$(\alpha_1)^{-\frac{1}{4}}, \quad (\alpha_2)^{-\frac{1}{4}}, \quad (\alpha_3)^{-\frac{1}{4}}.$$

Such a transformation is called a *pure strain* and the common axes of Q and Q_1 are called the *axes of the pure strain*.

If p is transformed into P their coordinates are connected by the equations (5). But in a pure strain $a^r_s x^s$ is in the direction of the perpendicular to the polar plane of p with respect to Q_1 . Therefore the transformed vector OP in a pure strain is along the normal to the polar plane of p with respect to Q_1 .

EXAMPLES.

1. Show that the general transformation (5) consists of a pure strain together with a rotation of one set of orthogonal axes to another.

2. If there exists an orthogonal system of axes in which the mixed tensor a^r_s has all its components, for which r and s are unequal, zero, show that the associated tensor a_{rs} is symmetric and conversely.

3. If a_{rs} is a symmetric tensor, show that the transformation (5) is a pure strain.

4. Prove the following construction for the point P when p is given: Let Π be the foot of the perpendicular from O on the polar plane of p with respect to Q_1 , then P is the inverse of Π with respect to the unit sphere with centre O .

5. In order that a transformation may be a uniform dilatation from O , show that it is necessary and sufficient that $a^r_s = k \delta_s^r$.

4. Rigid body displacements.

We shall say that the transformation (5) is a rigid body transformation when the distance between every pair of points remains unaltered under the transformation. Also the origin is a fixed point and hence the distance of any point from the origin is unaltered by the transformation. Thus we must have

$$g_{mn} X^m X^n = g_{mn} x^m x^n. \quad (19)$$

Moreover if this equation is satisfied for every point and its transform, then the distance between every pair of points is unaltered. Hence equation (19), true for all values of x^r , is the necessary and sufficient condition that a displacement be one of a rigid body. The condition is therefore

$$g_{mn} a_{\cdot r}^m a_{\cdot s}^n = g_{rs}. \quad (20)$$

This may also be seen geometrically from the fact that the quadric Q must be a sphere.

Now

$$|g_{mn} a_{\cdot r}^m a_{\cdot s}^n| = |g_{mn} a_{\cdot r}^m| \cdot |a_{\cdot s}^n| = |g_{mn}| |a_{\cdot s}^r|^2.$$

Hence equations (20) give

$$|a_{\cdot s}^r|^2 = 1, \quad (21)$$

and $|a_{\cdot s}^r|$ is either plus or minus one. If we suppose the rigid body transformation to be a continuous one, then (21) holds all the time, whilst initially $a_{\cdot s}^r = \delta_s^r$. That is, $|a_{\cdot s}^r|$, which must be continuous, is initially $+1$ and its square always equals $+1$. Therefore we must have

$$|a_{\cdot s}^r| = 1. \quad (22)$$

Again, let us inquire if any points remain invariable under the rigid body motion. Such points must satisfy the equations

$$(a_{\cdot s}^r - \delta_s^r) x^s = 0. \quad (23)$$

That this may be satisfied for non-zero values of x^r we must have

$$|a_{\cdot s}^r - \delta_s^r| = 0.$$

We have

$$\begin{aligned} g |a_{\cdot n}^m| |a_{\cdot s}^r - \delta_s^r| &= |g_{mn} a_{\cdot r}^m (a_{\cdot s}^n - \delta_s^n)| \\ &= |g_{rs} - g_{ms} a_{\cdot r}^m| = g |\delta_s^r - a_{\cdot s}^r|. \end{aligned}$$

Hence

$$|a_{\cdot s}^r - \delta_s^r| = |\delta_s^r - a_{\cdot s}^r|.$$

But this determinant is of the third order and therefore a change of sign in each constituent changes the sign of the determinant. Therefore

$$|a_{\cdot s}^r - \delta_s^r| = 0, \quad (24)$$

and (23) is true for certain non-zero values of x^r . If these equations are true for a given point x_0^r they are also true for all points on the line joining it to the origin. Consequently there is a line of invariable points through the origin.

Let us choose our coordinate axes \bar{x}^r so that they are orthogonal Cartesian and this line becomes the \bar{x}^3 -axis. Then (23) becomes

$$\bar{a}_{,3}^r = \delta_a^r.$$

Equations (20) for $s = 3$ give

$$\bar{a}_{,r}^3 = \delta_r^3.$$

Hence the transformation is

$$\bar{X}^1 = \bar{a}_{,1}^1 \bar{x}^1 + \bar{a}_{,2}^1 \bar{x}^2, \quad \bar{X}^2 = \bar{a}_{,1}^2 \bar{x}^1 + \bar{a}_{,2}^2 \bar{x}^2, \quad \bar{X}^3 = \bar{x}^3.$$

Also the remaining equations (20) are now

$$\begin{aligned} (\bar{a}_{,1}^1)^2 + (\bar{a}_{,1}^2)^2 &= 1, \\ \bar{a}_{,1}^1 \bar{a}_{,2}^1 + \bar{a}_{,1}^2 \bar{a}_{,2}^2 &= 0, \\ (\bar{a}_{,2}^1)^2 + (\bar{a}_{,2}^2)^2 &= 1, \end{aligned}$$

and if we put, $\bar{a}_{,1}^1 = \cos \alpha$, $\bar{a}_{,1}^2 = \sin \alpha$, we deduce with the help of (22) that

$$\bar{a}_{,2}^1 = -\sin \alpha, \quad \bar{a}_{,2}^2 = \cos \alpha.$$

Thus the transformation is

$$\bar{X}^1 = \bar{x}^1 \cos \alpha - \bar{x}^2 \sin \alpha, \quad \bar{X}^2 = \bar{x}^1 \sin \alpha + \bar{x}^2 \cos \alpha, \quad \bar{X}^3 = \bar{x}^3,$$

which is a rotation through an angle α round the \bar{x}^3 -axis. Also the invariant $a_{,r}^r$ equals $(\bar{a}_{,1}^1 + \bar{a}_{,2}^2 + \bar{a}_{,3}^3)$, that is,

$$a_{,r}^r = 1 + 2 \cos \alpha. \quad (25)$$

We therefore find that the transformation (5), p. 121, where the a 's satisfy the relations (20), is a rotation through an angle α round a direction λ^r where

$$\alpha = \cos^{-1} \frac{1}{2} (a_{,r}^r - 1), \quad (26)$$

and λ^r is the solution of the equations

$$(a_{,s}^r - \delta_s^r) \lambda^s = 0. \quad (27)$$

There cannot be more than one solution of these equations unless $\alpha = 0$, that is, unless $a_{,s}^r = \delta_s^r$ and the transformation consists merely of the identical relations $X^r = x^r$.

EXAMPLES.

1. Show that in a rigid body motion the roots of the equation

$$|a_{,s}^r - \theta \delta_s^r| = 0 \quad \text{are } (1, e^{i\alpha}, e^{-i\alpha}).$$

2. If the rotation is of amount α round a line through O in the direction λ^r , show that the a 's are given by

$$a_{,s}^r = \cos \alpha \delta_s^r + (1 - \cos \alpha) \lambda^r \lambda_s - \varepsilon^{rmn} g_{ms} \lambda_n \sin \alpha.$$

3. Defining a half-turn as a rotation through two right angles, show that a half-turn is given by

$$\alpha'_s = 2 \lambda^r \lambda_s - \delta_s^r.$$

4. Show that two successive rotations round a point are equivalent to a single rotation.

5. Infinitesimal deformations.

Let us next consider a *small* homogeneous deformation, so that the point X^r is near the original point x^r . Thus if we put

$$\alpha'_s = \delta_s^r + b^r_{s}, \dots \dots \dots (28)$$

the quantities b^r_{s} are infinitesimals of the first order. We shall neglect all quantities of the second or higher orders.

Putting $b^r_{s} = g_{rs} b^r_s$, we shall use the notation

$$\varepsilon_{rs} = \frac{1}{2}(b_{rs} + b_{sr}), \quad \omega_{rs} = \frac{1}{2}(b_{rs} - b_{sr}) \quad \dots (29)$$

so that ε_{rs} is a *symmetric double tensor* and ω_{rs} a *skew-symmetric tensor*. Also

$$b_{rs} = \varepsilon_{rs} + \omega_{rs},$$

and hence the transformation is

$$X^r = x^r + (\varepsilon^r_s + \omega^r_s) x^s,$$

or, writing $X^r - x^r = \delta x^r$,

$$\delta x^r = (\varepsilon^r_s + \omega^r_s) x^s \quad \dots \dots \dots (30)$$

If we take successively the infinitesimal deformations

$$\begin{aligned} X_1^r &= x^r + \varepsilon^r_s x^s \\ X_2^r &= x^r + \omega^r_s x^s, \quad \dots \dots \dots (31) \end{aligned}$$

we find that the combined effect is

$$X^r = X_2^r + \varepsilon^r_s X_2^r = x^r + \varepsilon^r_s x^s + \omega^r_s x^s,$$

when we neglect quantities of the second order. Thus the successive transformations (31) give (30) and the transformations may be taken in any order. We are therefore led to discuss separately the transformations

$$\delta x^r = \varepsilon^r_s x^s, \quad \dots \dots \dots (32)$$

$$\delta x^r = \omega^r_s x^s, \quad \dots \dots \dots (33)$$

where ε_{rs} is symmetric and ω_{rs} is skew-symmetric.

Let us first consider (32). We see from § 3, p. 123, that it is a *pure strain*. The quadric Q becomes

$$(g_{mn} + 2\varepsilon_{mn})x^m x^n = 1,$$

and the quadric Q_1 is

$$(g_{mn} + \varepsilon_{mn})x^m x^n = 1.$$

Thus they are both connected with the quadric S whose equation is

$$\varepsilon_{mn}x^m x^n = 1. \quad (34)$$

In fact, $Q = 2S + \Sigma$, $Q_1 = S + \Sigma$, where $\Sigma = 1$ is the equation of the unit sphere. The quadric (34) is called the *strain quadric*, and we see that the displacement δx^r of the point x^r is along the perpendicular to the polar plane of x^r with respect to S and the magnitude of the displacement is the reciprocal of the perpendicular from O on this polar plane.

Let us now consider (33). We easily see from § 4, p. 124, that (33) represents an infinitesimal rotation round a certain line through O , or we may see this as follows. We know that ω is skew-symmetric. Let us define the vector

$$\omega = -\frac{1}{2}\varepsilon^{rst}\omega_{st}, \quad (35)$$

and let us choose rectangular Cartesian axes \bar{x}^r such that this vector is along the \bar{x}^3 -axis. In such coordinates all the components of ω_{rs} vanish except the two $\bar{\omega}_{12}$, $\bar{\omega}_{21}$, which are equal in magnitude but opposite in sign. We then find

$$\delta \bar{x}^1 = -\delta\alpha \bar{x}^2, \quad \delta \bar{x}^2 = \delta\alpha \bar{x}^1, \quad \delta \bar{x}^3 = 0, \quad (36)$$

where we have put $\bar{\omega}_{12} = -\bar{\omega}_{21} = -\delta\alpha$. This is an infinitesimal rotation round the \bar{x}^3 -axis and of amount $\delta\alpha$. We can easily verify that the magnitude of ω^r is $\delta\alpha$. Hence the vector ω^r not only gives the direction of the axis of rotation but its magnitude gives the angle of rotation. The rotation is therefore completely defined by ω^r .

We conclude that the most general infinitesimal transformation (30) is obtained by two successive infinitesimal transformations, one a pure strain and the other a rotation.

EXAMPLES.

1. Show that, if $\delta^r_s + b^r_s$ is to be an infinitesimal rotation, then b_{rs} must be skew-symmetric.

2. Show that two infinitesimal rotations round O , defined by the vectors ω^r_1 and ω^r_2 , are together equivalent to a rotation round O defined

by $\omega^r = \omega_1^r + \omega_2^r$, and the order of the rotations is immaterial. Thus the combined effect of two infinitesimal rotations is given by the vector sum of their vectors and infinitesimal rotations can be treated as vectors in their combination.

3. Two infinitesimal deformations are together equivalent to an infinitesimal deformation whose tensor is obtained by adding the respective tensors of the two original deformations.

EXAMPLES X.

1. Show that there always exists at least one line through O such that it transforms into itself under the transformation (5), p. 121. Such a direction we call an *invariable direction*.

[If λ^r is an invariable direction, $(a^r_s - \theta \delta^r_s) \lambda^s = 0$, where θ is a root of $|a^r_s - \theta \delta^r_s| = 0$.]

2. If there are two (three) distinct roots of $|a^r_s - \theta \delta^r_s| = 0$, show that there are two (three) independent invariable directions.

3. Show that no three coplanar directions can be invariable directions unless two of the roots of this equation are equal, and in this case all directions in the plane are invariable.

4. If there are three distinct non-coplanar invariable directions, find the form of the transformation referred to these directions as axes.

$$[X^1 = a^1_1 x^1, \quad X^2 = a^2_2 x^2, \quad X^3 = a^3_3 x^3].$$

5. Classify the different types of affine transformations (5), p. 121, according as there exist 3, 2 or 1 independent invariable directions.

6. If $|a^r_s|$ is zero, show that there exists a direction such that every point on the line through p in this direction is transformed into the same point P (the transform of p). Also show that if all the first minors of $|a^r_s|$ are not zero, there is only one such direction.

7. With the notation of § 2 (p. 121), show that (i) $|a^r_s| = \sqrt{\alpha_1 \alpha_2 \alpha_3}$, (ii) any volume v is transformed into a volume V where

$$V/v = \sqrt{\alpha_1 \alpha_2 \alpha_3} = |a^r_s|.$$

8. *A shear. All points in a certain plane are fixed and all points in any parallel plane are displaced parallel to a fixed direction of the given plane a distance which is proportional to the distance between the planes. If μ^r is perpendicular to the given plane and λ^r parallel to the fixed direction, show that for a shear*

$$a^r_s = \delta^r_s + k \lambda^r \mu_s.$$

9. *Simple extension. All lines parallel to a given direction are extended in a fixed ratio and all lines perpendicular to this direction are invariable in length. Show that, if λ^r is the given direction,*

$$a^r_s = \delta^r_s + k \lambda^r \lambda_s.$$

10. If a^r_s represents a rotation round the direction λ^r , show that the tensor

$$\alpha_{mn} = \lambda_m \lambda_n - \cos \alpha (g_{mn} - \lambda_m \lambda_n)$$

is skew-symmetric.

[Calculate it when λ^r is the x^3 -axis and the coordinates are rectangular.]

11. If a_{rs}^r is a rotation and a_{mn} is symmetric show that the rotation must be through an even number of right angles.

12. If the rotation is not a half turn, the axis of rotation is given by the unit vector $\varepsilon^{rmn} a_{mn} / (-2 \sin \alpha)$.

13. Show that two half turns round parallel lines are equivalent to a simple translation in the direction perpendicular to both lines and equal to twice the distance between them.

[Use the formula of Ex. 3 (p. 126).]

14. Show that two successive half turns round intersecting lines is equivalent to a rotation round the perpendicular to these two directions and through an angle equal to twice the angle between the lines.

15. A rigid body is moving round a fixed point O , show that a vector ξ^r , which is fixed in the body, has its components, referred to fixed axes in space, changed in a time δt by an amount

$$\delta \xi^r = \omega_{rs}^r \xi^s,$$

ω_{rs} being skew-symmetric.

(i) Show that the *velocity* of any point of the body is given by

$$\dot{x}^r = \Omega_{st}^r x^s$$

where $\Omega_{st}^r = \lim_{\delta t \rightarrow 0} \omega_{st}^r / \delta t$.

[The vector $\Omega^r = -\frac{1}{2} \varepsilon^{rst} \Omega_{st}$ is called the *angular velocity vector*.]

(ii) Show that any vector fixed in space has its components, referred to axes fixed in the body, changed by an amount

$$\delta \xi^r = -\omega_{rs}^r \xi^s.$$

16. Show that the most general infinitesimal transformation of a rigid body is of the form

$$\delta x^r = \beta^r + \omega_{st}^r (x^s - x_0^s),$$

which is a translation β^r combined with a rotation round a line through the point x_0^r . Show that x_0^r can be chosen so that β^r is parallel to ω^r . In this case the line through x_0^r is called the *central axis* of the displacement.

PART III

Differential Geometry

CHAPTER XI

CURVILINEAR COORDINATES

1. General coordinate systems.

The position of a point in space is determined by its coordinates referred to an orthogonal Cartesian system of axes, and we shall denote such a system of coordinates by (y^1, y^2, y^3) . In Part II we took a *linear* transformation of the y 's and arrived at a system of rectilinear coordinates. Let us now take the more general functional transformation and let us examine what kind of coordinates we obtain.

We take the transformation

$$x^r = f^r(y^1, y^2, y^3), \quad (1)$$

where f^1, f^2, f^3 are arbitrary functions of the y 's, which we shall suppose possess derivatives up to any required order. Also it is well-known that if the determinant

$$\frac{\partial x^r}{\partial y^s} \equiv \frac{\partial (x^1, x^2, x^3)}{\partial (y^1, y^2, y^3)} \quad (2)$$

is different from zero, then the transformation (1) is reversible, that is, we can solve (1) for the y 's, giving

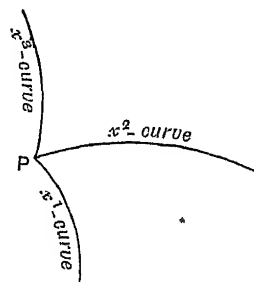
$$y^r = g^r(x^1, x^2, x^3). \quad (3)$$

From the formulae (1) and (3) we see that to each set of values y^r there is a unique set x^r and *vice versa*. Hence the variables x^r determine a point in space uniquely and we call them (for reasons which will very soon be apparent) *curvilinear coordinates of the point*.

We shall examine what geometrical significance these coordinates have. Now

$$f^1(y^1, y^2, y^3) = \text{const.} \quad (4)$$

is the equation of a surface and as the value of the constant varies we get a family of surfaces. Thus $x^1 = \text{const.}$ gives us a family of surfaces and to say that a point has x^1 as its first coordinate merely means that the point lies on a certain member of the family (4). Similarly $x^2 = \text{const.}$ and $x^3 = \text{const.}$ are the equations of two other families of surfaces and to say that a point has coordinates x^2, x^3 means that it lies on certain members of these two families. In other words, we are given three families of surfaces and the position of any point P is determined by giving the three members of these families of which it is the intersection. Moreover the condition that the determinant (2) does not vanish merely expresses the fact that the three surfaces obtained by taking a member of each family intersect in one and only one point, thus defining the position of that point uniquely.



The surfaces $x^1 = \text{const.}$, $x^2 = \text{const.}$, $x^3 = \text{const.}$ we call the *coordinate surfaces* and we shall refer to them briefly as the x^1 -surfaces, the x^2 -surfaces and the x^3 -surfaces respectively. Also the intersections of these surfaces give us three curves through every point P , two of them lying on each coordinate surface. These curves we shall call the *coordinate curves*. We see that along the curve which is the intersection of the x^2 -surface and the x^3 -surface x^1 alone varies, the other coordinates being constants, and we shall refer to this curve as the x^1 -curve. Similarly we shall call the other coordinate curves the x^2 -curve and the x^3 -curve. The coordinate curves in this system of coordinates are thus seen to be *curved* lines and the name "curvilinear coordinates" is now justified.

If we take any other system of curvilinear coordinates \bar{x}^r , they are also connected with the y^r by formulae of the type (1) and (3). Hence x^r and \bar{x}^r must be related by formulae

$$\left. \begin{aligned} \bar{x}^r &= \bar{x}^r(x^1, x^2, x^3) \\ x^r &= x^r(\bar{x}^1, \bar{x}^2, \bar{x}^3) \end{aligned} \right\}, \quad \dots \quad (5)$$

that is, the transformation from one set of curvilinear coordinates to another is a functional transformation. Moreover the determinant

$$\left| \frac{\partial \bar{x}^r}{\partial x^s} \right| = \left| \frac{\partial \bar{x}^r}{\partial y^m} \right| \cdot \left| \frac{\partial y^m}{\partial x^s} \right|$$

is different from zero. It is obvious that rectilinear coordinates are particular cases of curvilinear coordinates, namely, the cases where the functions f^r in (1) are linear functions.

EXAMPLES.

Verify that the following curvilinear systems have the properties mentioned:

1. *Spherical Polar Coordinates.* These are defined by

$$y^1 = x^1 \sin x^2 \cos x^3, \quad y^2 = x^1 \sin x^2 \sin x^3, \quad y^3 = x^1 \cos x^2.$$

The x^1 -surfaces are spheres centre O , the x^2 -surfaces are right circular cones, the x^3 -surfaces are planes through the y^3 -axis. Give the formulae (1) for this system.

2. *Cylindrical Polar Coordinates.*

$$y^1 = x^1 \cos x^2, \quad y^2 = x^1 \sin x^2, \quad y^3 = x^3.$$

The x^1 -surfaces are cylinders with the y^3 -axis as their common axis, the x^2 -surfaces are planes through the y^3 -axis and the x^3 -surfaces coincide with the y^3 -planes.

3. *Elliptic Coordinates.*

$$y^1 = \left\{ \frac{(x^1 - a)(x^2 - a)(x^3 - a)}{(b - a)(c - a)} \right\}^{\frac{1}{2}}, \quad y^2 = \left\{ \frac{(x^1 - b)(x^2 - b)(x^3 - b)}{(c - b)(a - b)} \right\}^{\frac{1}{2}},$$

$$y^3 = \left\{ \frac{(x^1 - c)(x^2 - c)(x^3 - c)}{(a - c)(b - c)} \right\}^{\frac{1}{2}}$$

where $a > b > c > 0$. Here the coordinates satisfy the inequalities $x^1 \leq a > x^2 \leq b > x^3 \leq c$. Show that the x^1 -surfaces are ellipsoids, the x^2 -surfaces are hyperboloids of one sheet, the x^3 -surfaces are hyperboloids of two sheets and all the quadrics belong to the family of confocals

$$(y^1)^2/(x - a) + (y^2)^2/(x - b) + (y^3)^2/(x - c) = 1.$$

4. *Parabolic Coordinates.*

$$y^1 = x^1 x^2 \cos x^3, \quad y^2 = x^1 x^2 \sin x^3, \quad y^3 = \frac{1}{2} [(x^1)^2 - (x^2)^2].$$

The x^1 -surfaces and the x^2 -surfaces are paraboloids of revolution and the x^3 -surfaces are planes through the y^3 -axis.

2. Tensor-Fields.

We have already discussed tensors with respect to general functional transformations at pp. 30—33. For example, the system a'_{st} is a relative tensor of weight M , contravariant in r and covariant in s, t , with respect to the transformation (5) if its transformed components \bar{a}'_{st} satisfy the relations

$$\bar{a}'_{st} = \left| \frac{\partial x^i}{\partial \bar{x}^j} \right|^M \frac{\partial \bar{x}^r}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^s} \frac{\partial x^p}{\partial \bar{x}^t} a'_{n p} \quad . \quad . \quad . \quad (6)$$

We must now mention a point which did not arise in the case of linear transformations. The coefficients $\frac{\partial \bar{x}^r}{\partial x^s}$ in (6) are functions of x^r and therefore also of \bar{x}^r . Consequently if a'_{st} and \bar{a}'_{st} are two sets of quantities satisfying (6) for *one* point x^r , these equations will not in general be true at other points. In other words, a'_{st} is a tensor *at the point* x^r and is not in general a tensor at any other point, that is, *tensors are now localised at points*. Thus all the algebraical operations that we have described in Chapter II (p. 19) apply to tensors at a given point and not to tensors at different points, e. g., the operation of multiplying together two tensors to obtain a third must be performed on two tensors at the same point.

Again let us suppose that the a'_{st} are functions of x^r , and that the new components \bar{a}'_{st} , which are also functions of x^r , satisfy (6) *at every point where the functions are defined*. Then we are given a tensor at every point of a region of space and such an aggregate of tensors we call a *tensor-field*. In the sequel we shall be dealing mainly with tensor-fields and for brevity we shall refer to them merely as tensors, since there will be no confusion. Hence when we speak of a tensor being defined over a certain region of space we mean a tensor-field defined over that region. We have of course tensor-fields of different orders just as we have tensors of different orders. Thus we have *invariant* or *scalar-fields* and *vector-fields*.

The case of most importance for us is the case where $M = 0$ in (6), that is, where the tensors are *absolute*, and it is to be understood that all the tensors dealt with are absolute tensors unless the contrary is explicitly stated.

EXAMPLES.

1. Prove that

$$e_{rst} \quad \left| \frac{\partial x^i}{\partial \bar{x}^j} \right|^{-1} \quad \frac{\partial x^m}{\partial \bar{x}^r} \frac{\partial x^n}{\partial \bar{x}^s} \frac{\partial x^p}{\partial \bar{x}^t}, \quad \frac{\partial x^i}{\partial \bar{x}^r} \quad e^{mnp} \quad \frac{\partial \bar{x}^r}{\partial x^m} \frac{\partial \bar{x}^s}{\partial x^n} \frac{\partial \bar{x}^t}{\partial x^p}$$

and deduce that e_{rst} and e^{rst} are relative tensors of weights -1 and 1 respectively.

2. Prove that the Kronecker deltas are absolute tensors.

3. Show that the differentials dx^r form a contravariant tensor.4. If φ is an invariant function, show that $\frac{\partial \varphi}{\partial x^r}$ is a covariant vector.

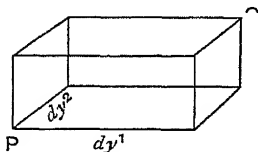
This vector is sometimes called the *gradient* of φ .

$$\left[\bar{\varphi} = \varphi \text{ and hence } \frac{\partial \bar{\varphi}}{\partial \bar{x}^r} = \frac{\partial \bar{\varphi}}{\partial x^s} \frac{\partial x^s}{\partial \bar{x}^r} = \frac{\partial \varphi}{\partial x^s} \frac{\partial x^s}{\partial \bar{x}^r} \right].$$

3. The line-element and the metric tensor. The ε -systems.

Let P be the point whose coordinates are x^r and let Q be a neighbouring point with coordinates $x^r + dx^r$. If we denote the infinitesimal distance PQ by ds , we call ds the *element of length* or the *line-element*, and we wish to find an expression for ds in terms of the differentials dx^r .

If we refer back to our orthogonal Cartesian system in which the coordinates of P are y^r and those of Q are $y^r + dy^r$, we see, from the elementary parallelepiped at P whose edges are dy^1, dy^2, dy^3 , that



$$ds^2 = (dy^1)^2 + (dy^2)^2 + (dy^3)^2 \\ = dy^i dy^i.$$

But $dy^i = \frac{\partial y^i}{\partial x^r} dx^r$.

Hence we see that

$$ds^2 = g_{mn} dx^m dx^n, \dots \dots \dots (7)$$

where we have put

$$g_n = \sum_{i=1}^3 \frac{\partial y^i}{\partial x^m} \frac{\partial y^i}{\partial x^n} = \frac{\partial y^i}{\partial x^m} \frac{\partial y^i}{\partial x^n} \dots \dots \dots (8)$$

These last equations show that g_{mn} is *symmetric*, and since ds is an invariant it follows from (7) that $g_{mn} dx^m dx^n$ is an invariant for arbitrary values of the contravariant vector dx^r . Consequently we conclude that g_{mn} is a *double covariant tensor* and we call it the *fundamental* or *metric tensor*.

If we denote by g the determinant $|g_{mn}|$ and by G^{mn} the cofactor of g_{mn} in g , we know (p. 15) that

$$3! g = e^{rst} e^{mnp} g_{rm} g_{sn} g_{tp}, \quad 2! G^{mn} = e^{mpq} e^{nrs} g_{pr} g_{qs} \quad (9)$$

But e^{rst} is a relative tensor of weight 1 (p. 30). Hence g is a relative invariant of weight 2, and G^{mn} is a relative tensor of weight 2. Moreover (8) gives us the result

$$g = |g_{mn}| = \left| \frac{\partial y^i}{\partial x^m} \right|^2,$$

so that g cannot vanish and is positive. Therefore, if we put

$$g^{mn} = \frac{G^{mn}}{g}, \quad (10)$$

that is, g^{mn} is the cofactor of g_{mn} in g , divided by g , we see that g^{mn} is an absolute contravariant tensor. Also we see that the quantities

$$\varepsilon_{rst} = \sqrt{g} e_{rst}, \quad \varepsilon^{rst} = \frac{1}{\sqrt{g}} e^{rst} \quad (11)$$

are absolute tensors and we shall call these systems the ε -systems.

When we are given a contravariant vector A^r , we can form an invariant A by means of the equation

$$A = (g_{mn} A^m A^n)^{\frac{1}{2}}, \quad (12)$$

and we call A the magnitude of the vector A^r . Similarly we define the magnitude of the covariant vector B_r by the formula

$$B = (g^{mn} B_m B_n)^{\frac{1}{2}}, \quad (13)$$

and we define a unit vector as one whose magnitude is unity. If we divide (7) by ds^2 we see immediately that

$$g_{mn} \frac{dx^m}{ds} \frac{dx^n}{ds} = 1 \quad (14)$$

and we conclude that $\frac{dx^r}{ds}$ is a unit vector.

We remark here that all that has been said at p. 44 regarding the lowering and raising of indices by means of g_{mn} and g^{mn} to obtain associated tensors applies without alteration in the present circumstances.

EXAMPLES.

1. Verify the expression of ds^2 in each of the coordinate systems (p. 132):

(i) *Orthogonal Cartesian.*

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

(ii) *Spherical Polar.*

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (x^1)^2 (\sin x^2)^2 (dx^3)^2.$$

(iii) *Cylindrical Polar.*

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (dx^3)^2.$$

(iv) *Elliptic.*

$$4 ds^2 = \frac{(x^3 - x^1)(x^2 - x^1)(dx^1)^2}{(x^1 - a)(x^1 - b)(x^1 - c)} + \frac{(x^1 - x^2)(x^3 - x^2)(dx^2)^2}{(x^2 - a)(x^2 - b)(x^2 - c)} \\ + \frac{(x^1 - x^3)(x^2 - x^3)(dx^3)^2}{(x^3 - a)(x^3 - b)(x^3 - c)}.$$

(v) *Parabolic.*

$$ds^2 = [(x^1)^2 + (x^2)^2][(dx^1)^2 + (dx^2)^2] + (x^1 x^2)^2 (dx^3)^2.$$

2. Prove that the magnitude of A^r is given by

$$A^2 = A^r A_r,$$

where A_r is the vector associated with A^r .

3. Show that the quadratic form $g_{mn} dx^m dx^n$ is positive definite.

4. Show that the vector A^r/A is a unit vector.

5. If the coordinate system reduces to a rectilinear system show that g_{mn} becomes the metric tensor of p. 41.

4. The angle between two directions.

If P is the point of coordinates x^r and Q is a neighbouring point of coordinates $x^r + dx^r$, we see that Q is determined uniquely by the differentials dx^r . But PQ obviously defines a certain direction in space and hence the differentials dx^r define a direction from P .

Let X^r be a contravariant vector whose magnitude is X . We can always find differentials dx^r in one coordinate system such that

$$dx^r = \varepsilon X^r, \quad \dots \dots \dots (15)$$

where ε is an infinitesimal positive factor, and the relation will hold in every coordinate system since it is a vector equation. Also using (7) and (12) we have

$$ds^2 = \varepsilon^2 g_{mn} X^m X^n = \varepsilon^2 X^2$$

that is, $\varepsilon = ds/X$. Hence (15) reads

$$\frac{dx^r}{ds} = \frac{X^r}{X} \cdot \cdot \cdot \cdot \cdot \cdot (16)$$

It is easy to see that X^r/X is a unit vector, which we may write λ^r , so that

$$X^r = X \lambda^r, \quad \frac{dx^r}{ds} = \lambda^r, \cdot \cdot \cdot \cdot \cdot (17)$$

and we conclude that *every unit vector λ^r defines a direction in space*. We see from the first of (17) that every contravariant vector consists of a magnitude and a unit vector and thus it also defines a direction which we call the direction of the vector.

If we are given two directions at P defined by the unit vectors λ^r, μ^r , let us consider the invariant $g_{mn} \lambda^m \mu^n$. When the system reduces to a rectilinear system we have seen at p. 43, that this invariant becomes $\cos \theta$ where θ is the angle between the directions. Therefore *the angle θ between the directions* is given by

$$\cos \theta = g_{mn} \lambda^m \mu^n \cdot \cdot \cdot \cdot \cdot (18)$$

Also if A^r, B^r are two vectors whose magnitudes are A and B , we see that the *scalar product* of A^r, B^r is

$$g_{mn} A^m B^n = AB \cos \theta, \cdot \cdot \cdot \cdot \cdot (19)$$

θ being the angle between the directions of A^r and B^r . In particular, the condition that the directions λ^r, μ^r should be *orthogonal* is

$$g_{mn} \lambda^m \mu^n = 0 \cdot \cdot \cdot \cdot \cdot (20)$$

We shall leave the student to prove, by taking a special coordinate system, that the *vector product* of A^r, B^r satisfies the equations

$$\varepsilon^{rmn} A_m B_n = AB \sin \theta \nu^r,$$

where ν^r is the unit vector perpendicular to both A^r and B^r . The sense of ν^r can be determined in a manner exactly analogous to that explained at p. 47.

EXAMPLES.

1. If $\varepsilon_{(1)}^r, \varepsilon_{(2)}^r, \varepsilon_{(3)}^r$ are the unit vectors along the directions of the tangents to the coordinate curves at P , show that they have the components

$$\varepsilon_{(1)}^r = \frac{1}{\sqrt{g_{11}}} \delta_1^r, \quad \varepsilon_{(2)}^r = \frac{1}{\sqrt{g_{22}}} \delta_2^r, \quad \varepsilon_{(3)}^r = \frac{1}{\sqrt{g_{33}}} \delta_3^r.$$

2. If $\theta_{12}, \theta_{23}, \theta_{31}$ are the angles between the coordinate curves at P , show that

$$\cos \theta_{12} = \frac{g_{12}}{\sqrt{g_{11} g_{22}}}, \quad \cos \theta_{23} = \frac{g_{23}}{\sqrt{g_{22} g_{33}}}, \quad \cos \theta_{31} = \frac{g_{31}}{\sqrt{g_{33} g_{11}}}.$$

3. Show that if the coordinate curves are mutually orthogonal, then the coordinate surfaces are mutually orthogonal. The coordinates are then said to be *orthogonal curvilinear coordinates*. Show that the necessary and sufficient conditions for this are that $g_{12} = g_{23} = g_{31} = 0$ at every point of space.

4. Show that the coordinates of Ex. 1, p. 136, are all orthogonal coordinates.

EXAMPLES XI.

1. Deduce from (8), p. 134, that $\frac{\partial y^i}{\partial x^m} = g_{mn} \frac{\partial x^n}{\partial y^i}$.

[Multiply (8) by $\frac{\partial x^n}{\partial y^j}$ and use the fact that $\frac{\partial y^i}{\partial x^n} \frac{\partial x^n}{\partial y^j} = \delta_j^i$.]

2. Prove that $\frac{\partial x^r}{\partial y^i} = g^{rm} \frac{\partial y^i}{\partial x^m}$ and deduce that $g^{mn} = \frac{\partial x^m}{\partial y^i} \frac{\partial x^n}{\partial y^i}$.

3. If λ^r, μ^r, ν^r are three unit vectors, show that

$$\varepsilon_{rst} \lambda^r \mu^s \nu^t = \sin \gamma \sin \theta,$$

where γ is the angle between μ^r, ν^r and θ is the angle which the direction λ^r makes with the plane containing μ^r, ν^r .

[Take special axes to evaluate the invariant.]

4. If $\varepsilon_{rst} \lambda^r \mu^s \nu^t = 0$, show that there exist quantities α, β such that $\nu^r = \alpha \lambda^r + \beta \mu^r$, that is, the directions are coplanar.

5. Show that $\varepsilon_{rst} \lambda^r \mu^s \nu^t$ is positive if the vectors λ^r, μ^r, ν^r can be deformed without becoming coplanar so that they coincide with the positive directions of the coordinate curves. The triad $(\lambda^r, \mu^r, \nu^r)$ is then said to have a *positive orientation*.

[See p. 47.]

6. Show that $\varepsilon_{rmn} \mu^m \nu^n / \sin \theta$, when θ is the angle between μ^r, ν^r , are the covariant components of the unit vector orthogonal to μ^r and ν^r .

7. Deduce from Ex. 6 that the covariant components of the unit vector orthogonal to the x^3 -surface are

$$0, 0, \frac{\sqrt{g}}{\sqrt{g_{11} g_{22} \sin \theta_{12}}}.$$

8. Show that the lengths of small elements of arc along the coordinate curves are

$$ds_1 = \sqrt{g_{11}} dx^1, \quad ds_2 = \sqrt{g_{22}} dx^2, \quad ds_3 = \sqrt{g_{33}} dx^3.$$

9. *Element of Volume.* Show that the element of volume dV is given by

$$dV = \sqrt{g} dx^1 dx^2 dx^3.$$

[If small lengths ds_1, ds_2, ds_3 are taken along the coordinate curves, we see from Ex. 3 that the volume of the small parallelepiped formed by them is $dV = ds_1 ds_2 ds_3 \epsilon_{rst} \epsilon_{(1)}^r \epsilon_{(2)}^s \epsilon_{(3)}^t = \sqrt{g} ds_1 ds_2 ds_3 / \sqrt{g_{11} g_{22} g_{33}}$. Then use Ex. 8.]

10. If the curvilinear coordinates are *orthogonal*, show that

$$g = g_{11} g_{22} g_{33}, \quad g^{11} = \frac{1}{g_{11}}, \quad g^{22} = \frac{1}{g_{22}}, \quad g^{33} = \frac{1}{g_{33}},$$

$$g_{mn} = g^{mn} = 0, \quad (m \neq n).$$

11. If λ^r is a unit vector, show that the cosines of the angles which its direction makes with the coordinate curves are

$$\lambda_1 / \sqrt{g_{11}}, \quad \lambda_2 / \sqrt{g_{22}}, \quad \lambda_3 / \sqrt{g_{33}}.$$

12. If $\varphi = \text{const.}$ is the equation of a surface, show that the unit normal vector to the surface is given by

$$g^{rn} \frac{\partial \varphi}{\partial x^m} \left/ \left(g^{mn} \frac{\partial \varphi}{\partial x^m} \frac{\partial \varphi}{\partial x^n} \right)^{\frac{1}{2}} \right.$$

[For every direction dx^r on the surface we have $\frac{\partial \varphi}{\partial x^r} dx^r = 0$.]

13. Show that the angle between two surfaces $\varphi = \text{const.}$ and $\psi = \text{const.}$ is given by

$$\cos \theta = g^{mn} \frac{\partial \varphi}{\partial x^m} \frac{\partial \psi}{\partial x^n} \left/ \left(g^{mn} \varphi_m \varphi_n g^{pq} \psi_p \psi_q \right)^{\frac{1}{2}} \right.$$

14. Deduce that the angle φ_{12} between the coordinate surfaces $x^1 = \text{const.}$ and $x^2 = \text{const.}$ is

$$\cos \varphi_{12} = g^{12} / \sqrt{g^{11} g^{22}}.$$

15. The condition that two surfaces $\varphi = \text{const.}$ and $\psi = \text{const.}$ cut orthogonally is

$$g^{mn} \frac{\partial \varphi}{\partial x^m} \frac{\partial \psi}{\partial x^n} = 0.$$

16. If A_s^r is an absolute tensor, show that $\sqrt{g} A_s^r$ is a relative tensor of weight 1. It is sometimes called a *tensor density*.

CHAPTER XII

COVARIANT DIFFERENTIATION

1. A parallel field of vectors. The Christoffel symbols.

A curve in space is defined as the locus of a point whose coordinates depend on a single parameter. Let AB be a given curve and let the coordinates of any point P on AB be functions of the parameter t . If we take any vector at a given point of the curve and at every other point of AB take the vector equal to it in magnitude and parallel to it in direction, we shall obtain a vector X^r defined at each point of the curve and the components of X^r will be functions of t . In other words, we have a *parallel field of vectors along AB* , and our object is to find the equations which such a vector-field must satisfy.

Let us return to our original Cartesian coordinates y^r and let Y^r be the components of the vector-field in this coordinate system. Since the components of parallel vectors are equal in Cartesian systems, we see that the Y^r are all constants along the curve and consequently the derivatives of Y^r with respect to t are zero. Now

$$Y^i = X^m \frac{\partial y^i}{\partial x^m}.$$

Therefore, differentiating with respect to t , we have

$$\frac{dX^m}{dt} \frac{\partial y^i}{\partial x^m} + X^m \frac{\partial^2 y^i}{\partial x^m \partial x^n} \frac{dx^n}{dt} = \frac{dY^i}{dt} = 0.$$

If we multiply these equations by $g^{rp} \frac{\partial y^i}{\partial x^p}$ and sum i from 1 to 3, we obtain, from (8), p. 134,

$$\frac{dX^r}{dt} + g^{rp} \frac{\partial^2 y^i}{\partial x^m \partial x^n} \frac{\partial y^i}{\partial x^p} X^m \frac{dx^n}{dt} = 0. \quad \dots (1)$$

This leads us to consider the expression $\frac{\partial^2 y^i}{\partial x^m \partial x^n} \frac{\partial y^i}{\partial x^p}$. Referring again to (8), p. 134, we get, on differentiating these equations partially with respect to x^p ,

$$\frac{\partial g_{mn}}{\partial x^p} = \frac{\partial^2 y^i}{\partial x^m \partial x^p} \frac{\partial y^i}{\partial x^n} + \frac{\partial y^i}{\partial x^m} \frac{\partial^2 y^i}{\partial x^p \partial x^n} \dots \quad (2)$$

and of course these equations are true when m, n, p take any of the values 1, 2, 3. If we take the two equations obtained by permuting m, n, p cyclically in (2) and subtract (2) from their sum, we get

$$\frac{\partial g_{np}}{\partial x^m} + \frac{\partial g_{pm}}{\partial x^n} - \frac{\partial g_{mn}}{\partial x^p} = 2 \frac{\partial^2 y^i}{\partial x^m \partial x^n} \frac{\partial y^i}{\partial x^p} \dots \quad (3)$$

Hence if we write for brevity

$$[mn, p] = \frac{1}{2} \left(\frac{\partial g_{np}}{\partial x^m} + \frac{\partial g_{pm}}{\partial x^n} - \frac{\partial g_{mn}}{\partial x^p} \right) \dots \quad (4)$$

and substitute (3) in (1), we have

$$\frac{dX^r}{dt} + g^{rp}[mn, p] X^m \frac{dx^n}{dt} = 0.$$

If we further write

$$\left\{ \begin{matrix} r \\ mn \end{matrix} \right\} = g^{rp}[mn, p], \dots \quad (5)$$

we have the equations (1) in the final form

$$\frac{dX^r}{dt} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} X^m \frac{dx^n}{dt} = 0, \dots \quad (6)$$

and the parallel vector-field along AB must satisfy this differential equation.

It is well-known from the theory of differential equations that if a vector-field X^r satisfies (6) along AB , and has a given value at one point of AB , then the vector-field is unique and completely determined along AB . But we have seen that the field obtained by taking the parallels to the given vector is one solution of (6). It is hence the unique solution and we have proved the converse theorem that any field of vectors along AB satisfying (6) is a parallel vector-field along the curve.

The quantities $[mn, p]$ and $\left\{ \begin{matrix} r \\ mn \end{matrix} \right\}$, defined by (4) and (5), are of the greatest importance. They are called *the Christoffel symbols of the first and second kinds* respectively, or they are

sometimes referred to as the *three-index symbols*.* It is seen at once that they are symmetrical in m, n — an important property.

If we take a vector at a given point of space and take at every point of space the parallel vector, we shall in this way define a parallel vector-field X^r , whose components are functions of the coordinates x^r . Moreover if we take any curve issuing from a point P , the vectors of the field along the curve must satisfy (6). But we now have $\frac{dX^r}{dt} = \frac{\partial X^r}{\partial x^n} \frac{dx^n}{dt}$ since X^r is a function of the x 's, and (6) becomes

$$\left[\frac{\partial X^r}{\partial x^n} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} X^m \right] \frac{dx^n}{dt} = 0.$$

This relation must be true for all curves starting from P , that is, for all values of the vector $\frac{dx^n}{dt}$ at P . Consequently the parallel vector-field satisfies the equation

$$\frac{\partial X^r}{\partial x^s} + \left\{ \begin{matrix} r \\ ms \end{matrix} \right\} X^m = 0, \dots \dots \dots (7)$$

and the converse is also true.

EXAMPLES.

1. *To find the relations connecting the Christoffel symbols of two curvilinear coordinate systems.*

Let x^r and \bar{x}^r be the two curvilinear systems. We shall as usual distinguish by bars the quantities belonging to the second coordinate system. Let A^r be an arbitrary parallel vector-field defined over all space and let \bar{A}^r be its components in the new system. Since they are parallel vector fields we have from (7)

$$+ \left\{ \begin{matrix} r \\ ms \end{matrix} \right\} A^m = 0, \quad \frac{\partial A^r}{\partial \bar{x}^s} + \left\{ \begin{matrix} r \\ ms \end{matrix} \right\} \bar{A}^m = 0. \dots \dots (\alpha)$$

Now $\bar{A}^r = A^i \frac{\partial \bar{x}^r}{\partial x^i}$ and hence

* These symbols are very often written in the form $\left[\begin{matrix} mn \\ r \end{matrix} \right]$ and $\left\{ \begin{matrix} mn \\ r \end{matrix} \right\}$. We have adopted the modified notation above since it is more in conformity with our convention that a dummy index in a term should be once an upper index and once a lower index. It must be borne in mind, however, that the symbols are *not* tensors.

$$\frac{\partial \bar{A}^r}{\partial \bar{x}^s} = \frac{\partial A^i}{\partial x^j} \frac{\partial \bar{x}^r}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^s} + A^i \frac{\partial^2 \bar{x}^r}{\partial x^i \partial x^j} \frac{\partial x^j}{\partial \bar{x}^s}.$$

We deduce from (α) that

$$-\left\{ \begin{matrix} r \\ ms \end{matrix} \right\} \bar{A}^m = -\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{\partial \bar{x}^r}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^s} A^k + A^i \frac{\partial^2 \bar{x}^r}{\partial x^i \partial x^j} \frac{\partial x^j}{\partial \bar{x}^s}. \quad (\beta)$$

If we put $\bar{A}^m = A^k \frac{\partial \bar{x}^m}{\partial x^k}$ and multiply the equations (β) by $\frac{\partial \bar{x}^s}{\partial x^i}$, we have

$$\left[\frac{\partial^2 \bar{x}^r}{\partial x^k \partial x^i} + \left\{ \begin{matrix} r \\ st \end{matrix} \right\} \frac{\partial \bar{x}^s}{\partial x^k} \frac{\partial \bar{x}^t}{\partial x^i} - \left\{ \begin{matrix} i \\ kl \end{matrix} \right\} \frac{\partial \bar{x}^r}{\partial x^l} \right] A^k = 0.$$

This relation is true for all parallel fields A^r and therefore we have the final result

$$\frac{\partial^2 \bar{x}^r}{\partial x^k \partial x^i} = \left\{ \begin{matrix} i \\ kl \end{matrix} \right\} \frac{\partial \bar{x}^r}{\partial x^l} - \left\{ \begin{matrix} r \\ st \end{matrix} \right\} \frac{\partial \bar{x}^s}{\partial x^k} \frac{\partial \bar{x}^t}{\partial x^i}.$$

2. Prove the similar relation

$$\frac{\partial^2 x^i}{\partial \bar{x}^r \partial \bar{x}^s} = \left\{ \begin{matrix} i \\ rs \end{matrix} \right\} \frac{\partial x^i}{\partial \bar{x}^s} - \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{\partial x^j}{\partial \bar{x}^r} \frac{\partial x^k}{\partial \bar{x}^s}.$$

3. Show that $[mn, r]$ and $\{m_n\}$ are symmetric in m and n .

4. Deduce from Ex. 1 that the Christoffel symbols are not tensors.

5. Show that if the coordinates are rectilinear the Christoffel symbols are all zero.

[The g 's are then all constants.]

2. The intrinsic and covariant derivation of vectors.

We are now in a position to return to the problem which we mentioned at p. 33, namely, the problem of forming new tensors by the differentiation of given tensors. We confine ourselves first to invariants and vectors, and of course we are dealing only with absolute tensors.

The case of *invariants* is particularly simple. If φ is an invariant function of a parameter t , then in the new variables \bar{x}^r we have $\bar{\varphi} = \varphi$. But $\frac{d\varphi}{dt}$ is the limit of a ratio $\frac{\varphi(t+\delta t) - \varphi(t)}{\delta t}$ as δt tends to zero, and $\frac{d\bar{\varphi}}{dt}$ is the limit of an equal ratio. Consequently

$$\frac{d\bar{\varphi}}{dt} = \frac{d\varphi}{dt}, \quad \dots \dots \dots (8)$$

and we see that $\frac{d\varphi}{dt}$ is an *invariant*. Again if φ is an invariant function of x^r we have

$$\frac{\partial \overline{\varphi}}{\partial x^r} = \frac{\partial \varphi}{\partial x^m} \frac{\partial x^m}{\partial x^r} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (9)$$

which shows that $\frac{d\varphi}{dx^r}$ is a *covariant vector*. Thus the ordinary derivatives of invariants furnish us with invariants and vectors without further modification.

Let us next consider the case of a *covariant vector*. We consider a covariant vector X_r defined along a curve C , so that it is a function of a parameter t , and let us take an arbitrary contravariant vector A^r at a given point of the curve together with all its parallels along the curve. In other words A^r is a parallel field along C and satisfies (6), 'p. 141. At any point of C we know that $(X_r A^r)$ is an invariant, and therefore its derivative with respect to t is also an invariant. But

$$\frac{d}{dt}(X_r A^r) = \frac{dX_r}{dt} A^r + X_r \frac{dA^r}{dt} = \frac{dX_r}{dt} A^r - \left\{ \begin{matrix} r \\ m \ n \end{matrix} \right\} X_r A^m \frac{dx^n}{dt}.$$

Hence we conclude that

$$\frac{dX_r}{dt} - \left\{ \begin{matrix} m \\ r \ n \end{matrix} \right\} X_m \frac{dx^n}{dt} \Big| A^r$$

is an invariant. But A^r is an arbitrary parallel vector-field and it follows from the quotient law that

$$\frac{\delta X_r}{\delta t} \equiv \frac{dX_r}{dt} - \left\{ \begin{matrix} m \\ r \ n \end{matrix} \right\} X_m \frac{dx^n}{dt} \cdot \cdot \cdot \cdot \cdot \quad (10)$$

is a *covariant vector*. We call it the *intrinsic derivative* of X_r with respect to t and to distinguish it from the ordinary derivative we have denoted it by $\frac{\delta X_r}{\delta t}$. Thus we see that to obtain a new vector by differentiating X_r we must add to the ordinary derivative a number of terms containing the Christoffel symbols.

Let us consider the vector equation

$$\frac{\delta X_r}{\delta t} = \frac{dX_r}{dt} - \left\{ \begin{matrix} m \\ r \ n \end{matrix} \right\} X_m \frac{dx^n}{dt} = 0. \quad \cdot \cdot \cdot \cdot \cdot \quad (11)$$

If it is satisfied in any one coordinate system it is satisfied in every other. If the axes are Cartesian we have seen that the g_{mn} are constants and therefore the Christoffel symbols in such a system are all zero, that is, (11) reduces in this case to

$$\frac{dX_r}{dt} = 0;$$

from which it follows that the components of X_r in Cartesian coordinates are all constants. Consequently the X_r form a parallel field along C and (11) is the equation that a *covariant parallel vector-field* along C must satisfy.

We next take a contravariant vector-field X^r along C and we leave the student to prove in exactly the same manner, by taking an arbitrary covariant parallel field A_r , that

$$\frac{\delta X^r}{\delta t} \equiv \frac{dX^r}{dt} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} X^m \frac{dx^n}{dt} \dots \dots (12)$$

is a *contravariant vector*, which we call the *intrinsic derivative of X^r with respect to t* .

We pass on to the case where we have vector-fields defined as functions of x^r throughout space. Let X^r be such a contravariant vector-field, and let us take *any* curve (parameter t) issuing from a point P . We know that (12) is a contravariant vector. But

$$\frac{dX^r}{dt} = \frac{\partial X^r}{\partial x^s} \frac{dx^s}{dt}, \quad \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} X^m \frac{dx^n}{dt} = \left\{ \begin{matrix} r \\ ms \end{matrix} \right\} X^m \frac{dx^s}{dt},$$

and therefore

$$\left[\frac{\partial X^r}{\partial x^s} + \left\{ \begin{matrix} r \\ ms \end{matrix} \right\} X^m \right] \frac{dx^s}{dt}$$

is a contravariant vector. This is true for all curves starting from P , that is, for all values of $\frac{dx^s}{dt}$ at P , and hence (by the quotient law) the expression in square brackets is a *tensor*, contravariant in r and covariant in s . This tensor we call the *covariant derivative of X^r* and we denote it by $X^r_{,s}$, the comma before the index s indicating that the vector has been derived with respect to x^s . Thus

$$X^r_{,s} \equiv \frac{\partial X^r}{\partial x^s} + \left\{ \begin{matrix} r \\ ms \end{matrix} \right\} X^m. \dots \dots (13)$$

We prove in a similar manner that if X_r is a covariant vector-field defined throughout space

$$X_{r,s} \equiv \frac{\partial X_r}{\partial x^s} - \left\{ \begin{matrix} m \\ r s \end{matrix} \right\} X_m \dots \dots \dots (14)$$

is a tensor, covariant in both r and s , and we call it the *covariant derivative* of X_r .

EXAMPLES.

1. Prove that $\frac{d}{dt} (g_{mn} X^m X^n) = 2 g_{mn} X^m \frac{\delta X^n}{\delta t}$.
2. Prove that $\frac{d}{dt} (X^m Y_m) = \frac{\delta X^m}{\delta t} Y_m + X^m \frac{\delta Y_m}{\delta t}$.
3. Show that the equations of parallel vector fields are $\frac{\delta X^r}{\delta t} = 0$, $X^r_{,s} = 0$.

3. The intrinsic and covariant derivatives of tensors.

We shall now extend the results of the last section to the derivation of tensors of any order. Let X^r_{st} be a typical tensor of the third order, which is defined along a curve in terms of a parameter t . We already know (p. 33) that $\frac{dX^r_{st}}{dt}$ is *not* a tensor, and our problem is to find how we modify this derivative in order to obtain a tensor. Let us take *three arbitrary parallel vector-fields* A_r, B^r, C^r along the curve. Then at each point of the curve $X^r_{st} A_r B^s C^t$ is an invariant and consequently its derivative with respect to t is also an invariant. But

$$\begin{aligned} \frac{d}{dt} (X^r_{st} A_r B^s C^t) &= \frac{dX^r_{st}}{dt} A_r B^s C^t + X^r_{st} \frac{dA_r}{dt} B^s C^t \\ &\quad + X^r_{st} A_r \frac{dB^s}{dt} C^t + X^r_{st} A_r B^s \frac{dC^t}{dt}. \end{aligned}$$

The parallel vectors satisfy (6), p. 141, and (11), p. 144, and therefore, when we substitute from these equations the values of the derivatives of A_r, B^r, C^r , we obtain without difficulty

$$\frac{d}{dt}(X_{st}^r A_r B^s C^t) = \left[\frac{dX_{st}^r}{dt} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} X_{st}^m \frac{dx^n}{dt} \right. \quad X_{st}^r \frac{dx^n}{dt} \\ \left. \left\{ \begin{matrix} m \\ nt \end{matrix} \right\} X_{sm}^r \frac{dx^n}{dt} \right] A_r B^s C^t.$$

Since the right-hand side is an invariant and A_r, B^r, C^r are arbitrary vectors, we conclude from the quotient law that the expression inside the square brackets is a *tensor of the same type* as X_{st}^r . We denote it by $\frac{\delta X_{st}^r}{\delta t}$ and call it the *intrinsic derivative* of X_{st}^r with respect to t . Thus

$$\frac{\delta X_{st}^r}{\delta t} = \frac{dX_{st}^r}{dt} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} X_{st}^m \frac{dx^n}{dt} - \left\{ \begin{matrix} m \\ sn \end{matrix} \right\} X_{mt}^r \frac{dx^n}{dt} \\ - \left\{ \begin{matrix} m \\ nt \end{matrix} \right\} X_{sn}^r \frac{dx^n}{dt}. \quad ? \quad (15)$$

Again let us suppose that X_{st}^r is a tensor-field defined throughout space so that the components of the tensor are functions of x^r . If we take any curve passing through a point P we know that along this curve the expression (15) is a tensor. But $\frac{dX_{st}^r}{dt} = \frac{\partial X_{st}^r}{\partial x^u} \frac{dx^u}{dt}$, and we see that (15) can be written

$$\left[\frac{\partial X_{st}^r}{\partial x^u} + \left\{ \begin{matrix} r \\ mu \end{matrix} \right\} X_{st}^m - \left\{ \begin{matrix} m \\ su \end{matrix} \right\} X_{mt}^r - \left\{ \begin{matrix} m \\ tu \end{matrix} \right\} X_{sm}^r \right] \frac{dx^u}{dt},$$

which is therefore a tensor. This result is true for all curves passing through P , that is, for all values of $\frac{dx^r}{dt}$ at P . Hence the expression inside the square brackets is a *tensor with one covariant index more than X_{st}^r* . We call it the *covariant derivative* of X_{st}^r and denote it by $\bar{X}_{st,u}^r$, the comma again indicating differentiation with respect to x^u . Thus

$$\bar{X}_{st,u}^r \equiv \frac{\partial X_{st}^r}{\partial x^u} + \left\{ \begin{matrix} r \\ mu \end{matrix} \right\} X_{st}^m - \left\{ \begin{matrix} m \\ su \end{matrix} \right\} X_{mt}^r - \left\{ \begin{matrix} m \\ tu \end{matrix} \right\} X_{sm}^r. \quad (16)$$

The processes we have described are quite general and can be applied by the student to obtain the intrinsic and covariant derivatives of any type of tensor.

EXAMPLES.

1. Show that

$$\delta X_s^r = \frac{dX_s^r}{dt} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} X_s^m \frac{dx^n}{dt} - \left\{ \begin{matrix} m \\ sn \end{matrix} \right\} X_m^r \frac{dx^n}{dt}$$

$$X_{s,t}^r = \frac{\partial X_s^r}{\partial x^t} + \left\{ \begin{matrix} r \\ mt \end{matrix} \right\} X_s^m - \left\{ \begin{matrix} m \\ st \end{matrix} \right\} X_m^r.$$

2. Show that

$$X_{r,s} - X_{s,r} = \frac{\partial X_r}{\partial x^s} - \frac{\partial X_s}{\partial x^r}.$$

4. Conservation of the rules of the ordinary differential calculus. Ricci's lemma.

We already know from the ordinary differential calculus how to find the derivatives of the sums and products of functions, and we shall show that *the same rules apply to find the intrinsic and covariant derivatives of the sums and products of tensors.*

We first of all mark that if our coordinate system is rectilinear then the g_{mn} are all constants and the Christoffel symbols are identically zero. Consequently reference to (15) and (16) shows that in rectilinear systems the intrinsic derivative of a tensor is identical with its ordinary derivative and the covariant derivative is identical with the partial derivative. Therefore if we are given any relation connecting the ordinary and partial derivatives of tensors in rectilinear coordinates we have merely to substitute the intrinsic derivative for the ordinary and the covariant for the partial derivative in order to obtain the corresponding tensor relation which is true in all coordinate systems. For example, let us suppose that three tensor-fields A_s^r , B_s^r , C^t together with their derivatives are connected by the relation

$$\frac{dA_s^r}{dt} = \frac{\partial B_s^r}{\partial x^t} C^t$$

in a system of rectilinear coordinates. We see that the relation

$$\frac{\delta A_s^r}{\delta t} = B_{s,t}^r C^t,$$

which is a tensor relation, is true in a particular system of coordinates and is therefore true in every curvilinear system.

Let us consider the sum of two tensors A_s^r , B_s^r , which is defined by

$$C_s^r = A_s^r + B_s^r.$$

If these are functions of a parameter t we see that

$$\frac{dC_s^r}{dt} = \frac{dA_s^r}{dt} + \frac{dB_s^r}{dt},$$

and in particular this relation is true in any rectilinear system. We conclude therefore that

$$\frac{\partial C_s^r}{\partial t} = \frac{\partial A_s^r}{\partial t} + \frac{\partial B_s^r}{\partial t}, \quad (17)$$

and we prove in the very same way that

$$C_{s,t}^r = A_{s,t}^r + B_{s,t}^r, \quad (18)$$

so that these two formulae show that the ordinary rules for finding the derivatives of a sum also hold for intrinsic and covariant derivation.

We next consider the product of two tensors A_s^r , B_{st}^r and, to include every case, we may suppose the product is contracted. Thus

$$C_{st}^r = A_m^r B_{st}^m.$$

By ordinary differentiation we have

$$\frac{dC_{st}^r}{dt} = \frac{dA_m^r}{dt} B_{st}^m + A_m^r \frac{dB_{st}^m}{dt},$$

and in particular this relation is true in any rectilinear system. Hence we have the tensor relation

$$\frac{\partial C_{st}^r}{\partial t} = \frac{\partial A_m^r}{\partial t} B_{st}^m + A_m^r \frac{\partial B_{st}^m}{\partial t} \quad (19)$$

in every system, and similarly

$$C_{st,u}^r = A_{m,u}^r B_{st}^m + A_m^r B_{st,u}^m, \quad (20)$$

which show that the ordinary rules hold also for the products of tensors.

Having dealt with the rules of intrinsic and covariant differentiation of tensors, we come to some important theorems that must be mentioned in this connection. If we take the fundamental tensor g_{rs} , we have seen that its components

are all constants in rectilinear coordinates and therefore

$$\frac{\partial g_{rs}}{\partial x^t} = 0$$

in rectilinear systems. Consequently we must have in every coordinate system the tensor equation

$$g_{rs,t} = 0, \quad \dots \quad (21)$$

that is, *the covariant derivative of the fundamental tensor is zero*. This important result is known as Ricci's lemma, and we see that the same method proves that *the covariant derivative of the associated tensor g^{rs} is also zero*.

Again, the ε -systems and the Kronecker deltas are constants in any rectilinear system and we conclude that

$$\varepsilon_{rst,u} = \varepsilon_{\dots,u}^{rst} = \delta_{s,u}^r = \delta_{st,u}^{rp} = 0. \quad \dots \quad (22)$$

In other words, *the covariant derivatives of the ε -systems and the Kronecker deltas are all identically zero*. A direct consequence of these results is that *when we are finding the intrinsic or covariant derivative of any combination of tensors, the fundamental tensors, the ε -systems and the Kronecker deltas can all be regarded as constants for the purpose of differentiation*.

As an example, let us prove that two indices may be contracted either before or after the process of covariant differentiation has been performed. We first note that contracting two indices in a tensor is exactly equivalent to multiplying by δ_s^r and then contracting, e.g., $A_{rm}^m = \delta_q^p A_{rp}^q$. Therefore, since the Kronecker deltas may be regarded as constants for covariant differentiation,

$$(\delta_q^p A_{rp}^q)_{,s} = \delta_q^p (A_{rp}^q)_{,s}$$

and this proves our statement.

EXAMPLES.

1. Prove that $\frac{d}{dt}(g_{mn} X^m Y^n) = g_{mn} \frac{\delta X^m}{\delta t} Y^n + g_{mn} X^m \frac{\delta Y^n}{\delta t}$.
2. If $X_r = g_{rs} X^s$, show that $X_{r,t} = g_{rs} X^s_{,t}$ and deduce that if X^r is a parallel vector-field so is X_r .
3. Show that the raising and lowering of indices can be performed before or after the operations of covariant and intrinsic derivation.
4. Show that $\frac{\partial}{\partial x^r}(g_{mn} X^m Y^n) = X_{m,r} Y^m + X^m Y_{m,r}$.
5. Prove that if X is the magnitude of X^r , then $X_{,r} = X_{m,r} X^m / X$.

5. The divergence and rotor of a vector. The Laplacian.

If we are given a vector-field X^r we know that $X^r_{,s}$, its covariant derivative, is a mixed tensor. From this we can form the invariant

$$\Theta = X^r_{,r}, \quad \dots \dots \dots (23)$$

Let us see what value Θ takes in Cartesian coordinates. Now in Cartesians the covariant derivative becomes the ordinary partial derivative. Hence we have in a Cartesian system

$$\Theta = \frac{\partial X^1}{\partial x^1} + \frac{\partial X^2}{\partial x^2} + \frac{\partial X^3}{\partial x^3}.$$

This expression is called *the divergence of X^r* , which is therefore given in general coordinates by (23). If we take the associated vector X_r , it is obvious that Θ is also expressed by the equivalent formula

$$\Theta = g^{mn} X_{m,n} \dots \dots \dots (24)$$

which gives the divergence in terms of the covariant components of the vector.

Moreover if there exists an invariant function φ such that the covariant vector X_r is given by

$$X_r = \frac{\partial \varphi}{\partial x^r} = \varphi_{,r},$$

then X_r is called *the gradient of φ* . Also in this case the divergence of X_r is written $\Delta \varphi$, and we see from (24) that

$$\Delta \varphi = g^{mn} \varphi_{,mn} \dots \dots \dots (25)$$

In orthogonal Cartesian coordinates we easily see that this reduces to

$$\Delta \varphi = \frac{\partial^2 \varphi}{(\partial x^1)^2} + \frac{\partial^2 \varphi}{(\partial x^2)^2} + \frac{\partial^2 \varphi}{(\partial x^3)^2}$$

and this expression is called *the Laplacian of φ* . Hence the Laplacian of φ in general coordinates is given by (25).

Lastly, let us take the covariant derivative $X^r_{,s}$ of the covariant vector X_r and let us form the vector

$$R^r = \varepsilon^{rts} X_{s,t} \dots \dots \dots (26)$$

This vector is called *the rotor, or rotation, or curl of X_r* . To find its expression in a Cartesian coordinate system we remark that $\varepsilon^{rst} = e^{rst}$ in such a system and the covariant derivatives

become partial derivatives. Hence the components of R^r are now seen to be

$$\left(\frac{\partial X_3}{\partial x^2} - \frac{\partial X_2}{\partial x^3}\right), \quad \left(\frac{\partial X_1}{\partial x^3} - \frac{\partial X_3}{\partial x^1}\right), \quad \left(\frac{\partial X_2}{\partial x^1} - \frac{\partial X_1}{\partial x^2}\right).$$

It is to be noted that if X_r is the gradient of a function φ , then $X_{r,s} = X_{s,r}$ and we conclude that the rotor of a gradient is identically zero.

EXAMPLES.

1. Show that $[rt, s] + [st, r] = \frac{\partial g_{rs}}{\partial x^i}$.
2. Show that $[rs, t] = g_{mt} \left\{ \begin{matrix} m \\ rs \end{matrix} \right\}$.

6. The Riemann-Christoffel tensor. The Lamé relations.

We have seen how to obtain the covariant derivative of a tensor and since this is also a tensor we may proceed to take its covariant derivative. The resulting tensor we call the *second covariant derivative* of the original and we can obviously continue in this way to find covariant derivatives of any order.

Let us examine the second covariant derivative of a vector X_r . Taking the covariant derivative of (14), p. 146, we have, by (16), p. 147,

$$\begin{aligned} X_{r,st} &= \frac{\partial X_{r,s}}{\partial x^t} - \left\{ \begin{matrix} m \\ rt \end{matrix} \right\} X_{m,s} - \left\{ \begin{matrix} m \\ st \end{matrix} \right\} X_{r,m} \\ &= \frac{\partial^2 X_r}{\partial x^s \partial x^t} - \left\{ \begin{matrix} m \\ rs \end{matrix} \right\} \frac{\partial X_m}{\partial x^t} - \left\{ \begin{matrix} m \\ rt \end{matrix} \right\} \frac{\partial X_m}{\partial x^s} - \left\{ \begin{matrix} m \\ st \end{matrix} \right\} \frac{\partial X_r}{\partial x^m} \\ &\quad - X_m \left[\frac{\partial}{\partial x^t} \left\{ \begin{matrix} m \\ rs \end{matrix} \right\} - \left\{ \begin{matrix} m \\ rt \end{matrix} \right\} \left\{ \begin{matrix} p \\ st \end{matrix} \right\} - \left\{ \begin{matrix} m \\ st \end{matrix} \right\} \left\{ \begin{matrix} p \\ rt \end{matrix} \right\} \right]. \end{aligned}$$

Let us permute s and t in this formula and subtract the two expressions. Owing to the symmetrical properties of the Christoffel symbols we have

$$X_{r,st} - X_{r,ts} = R^p_{rst} X_p, \dots \dots \dots (27)$$

where we have put

$$R^p_{rst} = \frac{\partial}{\partial x^s} \left\{ \begin{matrix} p \\ rt \end{matrix} \right\} - \frac{\partial}{\partial x^t} \left\{ \begin{matrix} p \\ rs \end{matrix} \right\} + \left\{ \begin{matrix} m \\ rt \end{matrix} \right\} \left\{ \begin{matrix} p \\ ms \end{matrix} \right\} - \left\{ \begin{matrix} m \\ rs \end{matrix} \right\} \left\{ \begin{matrix} p \\ mt \end{matrix} \right\}. \quad (28)$$

Now the left-hand side of (27) is obviously a tensor and therefore the right-hand side is also a tensor. But X_p is an arbitrary vector and we conclude from the quotient law that $R^p{}_{rst}$ is a tensor. It is called the *Riemann-Christoffel tensor* and it will be noticed that it consists only of the g_{mn} and their derivatives up to the second order.

Let us lower the index p to obtain the associated tensor R_{prst} , that is,

$$R_{prst} = g_{pm} R^m{}_{rst},$$

and we shall now find an expression for this associated tensor. We have

$$\begin{aligned} g_{pm} \frac{\partial}{\partial x^s} \left\{ \begin{matrix} m \\ rt \end{matrix} \right\} &= \frac{\partial}{\partial x^s} \left(g_{pm} \left\{ \begin{matrix} m \\ rt \end{matrix} \right\} \right) - \left\{ \begin{matrix} m \\ rt \end{matrix} \right\} \frac{\partial g_{pm}}{\partial x^s} \\ &= \frac{\partial}{\partial x^s} [rt, p] - \left\{ \begin{matrix} m \\ rt \end{matrix} \right\} ([ps, m] + [ms, p]). \end{aligned}$$

But

$$\begin{aligned} R_{prst} &= g_{pm} R^m{}_{rst} = g_{pm} \frac{\partial}{\partial x^s} \left\{ \begin{matrix} m \\ rt \end{matrix} \right\} - g_{pm} \frac{\partial}{\partial x^s} \left\{ \begin{matrix} m \\ rs \end{matrix} \right\} \\ &\quad + \left\{ \begin{matrix} m \\ rt \end{matrix} \right\} [ms, p] - \left\{ \begin{matrix} m \\ rs \end{matrix} \right\} [mt, p]. \end{aligned}$$

Substituting the above results, we get

$$R_{prst} = \frac{\partial}{\partial x^s} [rt, p] - \frac{\partial}{\partial x^t} [rs, p] + \left\{ \begin{matrix} m \\ rs \end{matrix} \right\} [pt, m] - \left\{ \begin{matrix} m \\ rt \end{matrix} \right\} [ps, m]. \quad (29)$$

When we substitute the values of the Christoffel symbols (4), p. 141, in the first two terms this becomes

$$\begin{aligned} R_{prst} &= \frac{1}{2} \left(\frac{\partial^2 g_{pt}}{\partial x^r \partial x^s} + \frac{\partial^2 g_{rs}}{\partial x^p \partial x^t} - \frac{\partial^2 g_{ps}}{\partial x^r \partial x^t} - \frac{\partial^2 g_{rt}}{\partial x^p \partial x^s} \right) \\ &\quad + g^{mn} ([rs, m] [pt, n] - [rt, m] [ps, n]). \quad (30) \end{aligned}$$

We deduce from (30) that R_{prst} satisfies the relations

$$\left. \begin{aligned} R_{prst} &= -R_{rpst}, \\ R_{prst} &= -R_{prts}, \\ R_{prst} &= R_{stpr}. \end{aligned} \right\} \dots \dots \dots (31)$$

The first two of these equations express the fact that R_{prst} is skew-symmetric in p, r and also in s, t .

Let us put

$$S^{ij} = \frac{1}{4} \varepsilon^{ikl} \varepsilon^{jmn} R_{klmn}. \quad (32)$$

If we multiply this equation by $\varepsilon_{ipr} \varepsilon_{jst}$ and sum i, j from 1 to 3, then

$$\varepsilon_{ipr} \varepsilon_{jst} S^{ij} = \frac{1}{4} \delta_{pr} \delta_{st}^{\quad kl} R_{klmn} = \frac{1}{2} \delta_{st}^{\quad mn} R_{prmn} = R_{prst},$$

owing to the skew-symmetrical properties of R_{prst} . Thus

$$R_{prst} = \varepsilon_{ipr} \varepsilon_{jst} S^{ij}. \quad (33)$$

Moreover, using the last of equations (31),

$$\begin{aligned} S^{ij} &= \frac{1}{4} \varepsilon^{ikl} \varepsilon^{jmn} R_{klmn} = \frac{1}{4} \varepsilon^{ikl} \varepsilon^{jmn} R_{mnkl} \\ &= \frac{1}{4} \varepsilon^{jmn} \varepsilon^{ikl} R_{mnkl} = S^{ji}, \end{aligned}$$

that is, S^{ij} is a *symmetric* double tensor. Hence we conclude that in a space of three dimensions the Riemann-Christoffel tensor can be expressed in terms of a symmetric double tensor S^{ij} .

We next return to (27), and we examine what this tensor equation becomes in a Cartesian coordinate system. Since covariant derivatives become partial derivatives we see that the left-hand side becomes

$$\frac{\partial^2 X_r}{\partial x^s \partial x^i} - \frac{\partial^2 X_r}{\partial x^i \partial x^s}.$$

But this is identically zero since the order of partial differentiation is immaterial. Consequently the right-hand side of (27) is also zero for all vectors X_r and therefore

$$R^p_{rst} = 0, \quad (34)$$

that is, *the Riemann-Christoffel tensor is identically zero*.* It results also from (32) that

$$S^{ij} = 0, \quad (35)$$

and conversely if S^{ij} vanishes identically then (33) shows that R^p_{rst} does so too. The conditions (35) are six in number and these are six relations which must be satisfied identically by the fundamental tensor g_{mn} . We call them the *Lamé relations*.

EXAMPLE.

Show that $S^{11} = \frac{1}{g} R_{2323}$, $S^{33} = \frac{1}{g} R_{3112}$, etc.

* The reason why the Riemann-Christoffel tensor vanishes is that our space is Euclidean and so admits of Cartesian coordinate systems. In higher differential geometry we treat of more general spaces in which the Riemann-Christoffel tensor does not vanish.

EXAMPLES XII.

1. By writing out in full the tensor equation $g^{r..,t} = 0$, prove that

$$\frac{\partial g^{rs}}{\partial x^t} + g^{ms} \left\{ \begin{matrix} r \\ m t \end{matrix} \right\} + g^{mr} \left\{ \begin{matrix} s \\ m t \end{matrix} \right\} = 0.$$

2. By writing out in full the tensor equation $\varepsilon_{rst..n} = 0$ and putting $r, s, t = 1, 2, 3$, prove that

$$\frac{\partial \log \sqrt{g}}{\partial x^p} = \left\{ \begin{matrix} m \\ m p \end{matrix} \right\}.$$

3. Deduce from Exs. 1 and 2 that

$$\frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} g^{rs})}{\partial x^s} + \left\{ \begin{matrix} r \\ m n \end{matrix} \right\} g^{mn} = 0.$$

4. Show that the contravariant components of the rotor of X , in general coordinates are

$$\frac{1}{\sqrt{g}} \left(\frac{\partial X_3}{\partial x^2} - \frac{\partial X_2}{\partial x^3} \right), \quad \frac{1}{\sqrt{g}} \left(\frac{\partial X_1}{\partial x^3} - \frac{\partial X_3}{\partial x^1} \right), \quad \frac{1}{\sqrt{g}} \left(\frac{\partial X_2}{\partial x^1} - \frac{\partial X_1}{\partial x^2} \right).$$

5. Show that the divergence of X^r is given by

$$X^{r..,r} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r} (\sqrt{g} X^r).$$

[This is the formula for the divergence which is most useful for purposes of calculation.]

6. Show that the Laplacian of φ is given by

$$\Delta \varphi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r} \left(\sqrt{g} g^{rs} \frac{\partial \varphi}{\partial x^s} \right).$$

7. If X^{rs} is a contravariant tensor, show that

$$X^{r..,s} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^s} (\sqrt{g} X^{rs}) + X^{mn} \left\{ \begin{matrix} r \\ m n \end{matrix} \right\}$$

and show that the last term disappears if X^{rs} is skew-symmetric.

8. If R_{rs} is the contracted tensor $R^p_{..r..s..p}$, show that

$$R_{rs} = S_{rs} - S g_{rs},$$

where S_{rs} is the associated tensor of S^{rs} , and $S = g_{mn} S^{mn}$.

[Use the relation $g^{mn} \varepsilon_{m..p..r..n..s..t} = g_{ps} g_{rt} - g_{pt} g_{rs}$.]

9. *Orthogonal Curvilinear Coordinates.* Show that we have the following results when the coordinates are orthogonal:

$$(i) \quad g_{mn} = g^{mn} = 0 \quad (m \neq n), \quad g^{11} = \frac{1}{g_{11}}, \quad g^{22} = \frac{1}{g_{22}}, \quad g^{33} = \frac{1}{g_{33}}.$$

$$(ii) \quad \text{If we put } g_{11} = h_1^2, \quad g_{22} = h_2^2, \quad g_{33} = h_3^2,$$

$$\text{then } ds^2 = h_1^2 (dx^1)^2 + h_2^2 (dx^2)^2 + h_3^2 (dx^3)^2.$$

$$(iii) [ij, k] = 0, \quad [ij, i] = -[ii, j] = h_i \frac{\partial h_i}{\partial x^j}, \quad [ii, i] = h_i \frac{\partial h_i}{\partial x^i}.$$

$$(iv) \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = 0, \quad \left\{ \begin{matrix} j \\ ii \end{matrix} \right\} = -\frac{h_i}{h_i^2} \frac{\partial h_i}{\partial x^j}, \quad \left\{ \begin{matrix} i \\ ij \end{matrix} \right\} = \frac{\partial \log h_i}{\partial x^j} \quad \frac{\partial \log h_i}{\partial x^i}.$$

$$(v) \Delta \varphi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x^1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \varphi}{\partial x^1} \right) + \frac{\partial}{\partial x^2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \varphi}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \varphi}{\partial x^3} \right) \right].$$

$$(vi) S^{ij} = -\frac{1}{h_i^2 h_j h_k} \left[\frac{\partial}{\partial x^j} \left(\frac{1}{h_i} \frac{\partial h_k}{\partial x^i} \right) + \frac{\partial}{\partial x^k} \left(\frac{1}{h_i} \frac{\partial h_j}{\partial x^i} \right) + \frac{1}{h_i^2} \frac{\partial h_j}{\partial x^i} \frac{\partial h_k}{\partial x^i} \right],$$

$$S^{ij} = \frac{1}{h_i^2 h_j^2 h_k} \left[\frac{\partial^2 h_k}{\partial x^i \partial x^j} - \frac{\partial h_k}{\partial x^i} \frac{\partial \log h_i}{\partial x^j} - \frac{\partial h_k}{\partial x^j} \frac{\partial \log h_j}{\partial x^i} \right].$$

[Note that in the above formulae the summation convention has been dropped, and that in (iii), (iv), and (vi), i, j, k are all unequal.]

10. Write down the formulae of Ex. 9 for *spherical polar coordinates*. [Here $h_1 = 1$, $h_2 = x^1$, $h_3 = x^1 \sin x^2$ (p. 136).]

11. Write down the formulae of Ex. 9 for *cylindrical polar coordinates*. [Here $h_1 = 1$, $h_2 = x^1$, $h_3 = 1$ (p. 136).]

CHAPTER XIII

CURVES IN SPACE.

1. The tangent vector to a curve.

A space curve is the locus of a point whose coordinates are functions of a single parameter. Thus the coordinates of a point on a curve C are given by equations of the type

$$x^r = x^r(t) \dots \dots \dots (1)$$

From (7), p. 134, it is clear that the arc-length s along the curve satisfies the equation

$$(ds)^2 = g_{mn} \frac{dx^m}{dt} \frac{dx^n}{dt},$$

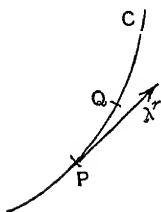
and consequently the arc s is given by the integral

$$= \int_{t_0}^t \left(g_{mn} \frac{dx^m}{dt} \frac{dx^n}{dt} \right)^{\frac{1}{2}} dt. \dots (2)$$

We shall usually take the arc s as the parameter along the curve, and we have the relation

$$g_{mn} \frac{dx^m}{ds} \frac{dx^n}{ds} = 1, \dots \dots \dots (3)$$

which shows that $\frac{dx^r}{ds}$ is a unit vector.



Let P be a point on the curve and let its coordinates be x^r . Q is a neighbouring point on C corresponding to an increment ds in the arc and its coordinates are $x^r + dx^r$. The vector $Lt \overrightarrow{PQ}/ds$ is called the *tangent vector* and we shall denote it by λ^r . Thus the tangent vector λ^r is given by

$$\frac{dx^r}{ds} = \lambda^r, \quad (4)$$

and we see from (3) that λ^r is the *unit tangent vector* of the curve C .

2. Normal vectors. The principal normal and binormal.

Any vector, which is orthogonal to the tangent vector, is said to be a normal vector of the curve. Hence the condition that a vector μ^r be normal to C is

$$g_{mn} \lambda^m \mu^n = 0. \quad (5)$$

Since λ^r is a unit vector,

$$g_{mn} \lambda^m \lambda^n = 1,$$

and if we take the intrinsic derivative (p. 148) of this equation, we get

$$g_n \lambda^m \frac{\delta \lambda^n}{\delta s} = 0.$$

This shows that the vector $\frac{\delta \lambda^r}{\delta s}$ is normal to the curve. Let us denote the *unit* vector co-directional with $\frac{\delta \lambda^r}{\delta s}$ by μ^r . Then

$$\mu^r = \frac{1}{\kappa} \frac{\delta \lambda^r}{\delta s}, \quad (\kappa > 0), \quad (6)$$

where κ is chosen to make μ^r a unit vector. The normal vector μ^r is called the *principal normal* of C and κ is called its *curvature* at the point in question.

Now let us take μ^r . Since it is a unit vector we prove, in exactly the same manner as for λ^r , that its intrinsic derivative $\frac{\delta \mu^r}{\delta s}$ is orthogonal to μ^r . Also if we differentiate (5) intrinsically with respect to s we have

$$g_{mn} \frac{\delta \lambda^m}{\delta s} \mu^n + g_{mn} \lambda^m \frac{\delta \mu^n}{\delta s} = 0,$$

that is,

$$g_{mn}\lambda^m\frac{\delta\mu^n}{\delta s} = -g_{mn}\frac{\delta\lambda^n}{\delta s}\mu^r = -\kappa g_{mn}\mu^r\mu^r = -\kappa.$$

This equation may be written in the equivalent form

$$g_{mn}\lambda^m\left(\frac{\delta\mu^n}{\delta s} + \kappa\lambda^n\right) = 0,$$

which shows that the vector $\left(\frac{\delta\mu^r}{\delta s} + \kappa\lambda^r\right)$ is orthogonal to λ^r .

But we have also the equation

$$g_{mn}\mu^m\left(\frac{\delta\mu^n}{\delta s} + \kappa\lambda^n\right) = 0,$$

showing that it is orthogonal to μ^r . Therefore the *unit* vector ν^r , defined by the equation

$$\nu^r = \frac{1}{\tau}\left(\frac{\delta\mu^r}{\delta s} + \kappa\lambda^r\right), \quad \dots (7)$$

is orthogonal both to λ^r and to μ^r . The quantity τ is chosen to make ν^r a unit vector and we see that λ^r, μ^r, ν^r form a mutually orthogonal triad of vectors. The sign of τ is not always positive but is so chosen that the invariant

$$\varepsilon_{ijk}\lambda^i\mu^j\nu^k = +1; \quad \dots \dots \dots (8)$$

in other words, the vectors $(\lambda^r, \mu^r, \nu^r)$ are to form a *triad with a positive orientation* (cf. p. 48).

The vector ν^r is called the *binormal* of C at the point, and τ is called the *torsion* of the curve.

EXAMPLES.

1. Prove that $\kappa = \left(g_{mn}\frac{\delta\lambda^m}{\delta s}\frac{\delta\lambda^n}{\delta s}\right)^{\frac{1}{2}}$.
2. Prove that $\tau = \varepsilon_{rst}\lambda^r\mu^s\frac{\delta\mu^t}{\delta s}$.
3. Show that $\nu^r = \varepsilon^{rmn}\lambda_m\mu_n$ where λ_r, μ_r are the associated vectors of λ^r and μ^r .
4. Show that $\mu_r = \frac{1}{\kappa}\frac{\delta\lambda_r}{\delta s}$, $\nu_r = \frac{1}{\tau}\left(\frac{\delta\mu_r}{\delta s} + \kappa\lambda_r\right)$.

3. The Frenet formulae.

Since ν^r is perpendicular to λ^r and μ^r and satisfies (8) we easily prove that

$$\nu^r = \varepsilon^{rmn} \lambda_m \mu_n.$$

Hence

$$\frac{\delta \nu^r}{\delta s} = \varepsilon^{rmn} \frac{\delta \lambda_m}{\delta s} \mu_n + \varepsilon^{rmn} \lambda_m \frac{\delta \mu_n}{\delta s} \dots \dots (9)$$

If we lower the indices in (6) and (7) and solve for $\frac{\delta \lambda_r}{\delta s}$ and $\frac{\delta \mu_r}{\delta s}$, we get

$$\frac{\delta \lambda_r}{\delta s} = \kappa \mu_r, \quad \frac{\delta \mu_r}{\delta s} = \tau \nu_r - \kappa \lambda_r.$$

Substituting these values in (9),

$$\frac{\delta \nu^r}{\delta s} = \varepsilon^{rmn} \kappa \mu_m \mu_n + \varepsilon^{rmn} \lambda_m (\tau \nu_n - \kappa \lambda_n) = \tau \varepsilon^{rmn} \lambda_m \nu_n = -\tau \mu^r.$$

We combine this last formula with (6) and (7), and we obtain the following relations:

$$\left. \begin{aligned} \frac{\delta \lambda^r}{\delta s} &= \kappa \mu^r, \\ \frac{\delta \mu^r}{\delta s} &= \tau \nu^r - \kappa \lambda^r, \\ \frac{\delta \nu^r}{\delta s} &= -\tau \mu^r. \end{aligned} \right\} \dots \dots \dots (10)$$

These formulae are called *the Frenet formulae of the curve* and we see that they are relations connecting κ , τ , λ^r , μ^r , ν^r .

EXAMPLES.

1. Using cylindrical polar coordinates prove that the curvature of the circle $x^1 = a$, $x^2 = t$, $x^3 = 0$ is $\frac{1}{a}$.

In cylindrical coordinates $ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (dx^3)^2$ and we can verify that the only non-zero Christoffel symbols are

$$\left\{ \begin{smallmatrix} 1 \\ 22 \end{smallmatrix} \right\} = -x^1, \quad \left\{ \begin{smallmatrix} 2 \\ 12 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 2 \\ 21 \end{smallmatrix} \right\} = \frac{1}{x^1}.$$

The tangent vector to the circle $x^1 = a$, $x^2 = t$, $x^3 = 0$ is

$$\lambda^r \equiv \frac{dx^r}{ds} = \left(0, \frac{dt}{ds}, 0 \right)$$

and it must satisfy $g_{mn}\lambda^m\lambda^n = 1$ at all points of the curve, that is,

$$g_{mn}\lambda^m\lambda^n = (x^1)^2 \left(\frac{dt}{ds}\right)^2 = a^2 \left(\frac{dt}{ds}\right)^2 = 1.$$

Consequently $\frac{dt}{ds} = \frac{1}{a}$. The first of the Frenet formulae give us

$$\kappa\mu^1 = \frac{\delta\lambda^1}{\delta s} = \frac{d\lambda^1}{ds} + \left\{ \begin{matrix} 1 \\ mn \end{matrix} \right\} \lambda^m \frac{dx^n}{ds} = \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} \lambda^2 \frac{dx^2}{ds} = -\frac{1}{a},$$

$$\kappa\mu^2 = \frac{\delta\lambda^2}{\delta s} = \frac{d\lambda^2}{ds} + \left\{ \begin{matrix} 2 \\ mn \end{matrix} \right\} \lambda^m \frac{dx^n}{ds} = \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} \lambda^2 \frac{dx^1}{ds} = 0,$$

$$\kappa\mu^3 = \frac{\delta\lambda^3}{\delta s} = \frac{d\lambda^3}{ds} + \left\{ \begin{matrix} 3 \\ mn \end{matrix} \right\} \lambda^m \frac{dx^n}{ds} = 0.$$

Also μ^r is a unit vector and so

$$\kappa^2 = g_{mn}(\kappa\mu^m)(\kappa\mu^n) = \frac{1}{a^2}.$$

We conclude that $\kappa = \frac{1}{a}$ and $\mu^r \equiv (-1, 0, 0)$.

2. In Ex. 1, show that $\tau = 0$ and $\nu^r \equiv (0, 0, 1)$.

3. Show that $\kappa\nu_r = \varepsilon_{rst}\lambda^s \frac{\delta\lambda^t}{\delta s}$.

4. Parallel vectors along a curve. The straight line.

If a vector X^r is defined along a curve then it must be given as a function of the arc s of the curve. Now we have seen at p. 141 that, if the vector X^r is a parallel field of vectors along a curve, it must satisfy the tensor equation

$$\frac{\delta X^r}{\delta s} = \frac{dX^r}{ds} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} X^m \frac{dx^n}{ds} = 0 \quad \dots \quad (11)$$

Also its covariant components X_r satisfy the corresponding tensor equation

$$\frac{\delta X_r}{\delta s} = \frac{dX_r}{ds} - \left\{ \begin{matrix} m \\ rn \end{matrix} \right\} X_m \frac{dx^n}{ds} = 0 \quad \dots \quad (12)$$

We can use this result to obtain immediately the equation of the straight line in general curvilinear coordinates. The tangent vector of a straight line is always in the same direction along the curve, that is, the tangent vector forms a parallel vector field and must satisfy (11). But the tangent vector is

$$\lambda \equiv \frac{dx^r}{ds}.$$

Hence the equations of a straight line are

$$\frac{\partial \lambda^r}{\partial s} = \frac{d^2 x^r}{ds^2} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \frac{dx^m}{ds} \frac{dx^n}{ds} = 0. \quad (13)$$

We may obtain these equations in another way by observing that (13) is a tensor equation and in Cartesian coordinates it reduces to $\frac{d^2 x^r}{ds^2} = 0$, which are the equations of a straight line in such coordinates.

Reference to the Frenet formulae shows that (13) merely expresses the fact that $\kappa = 0$. Hence we see that *the straight line has zero curvature*, and this property is characteristic of the straight line.

EXAMPLES.

1. If we put $\Phi = g_{mn} x'^m x'^n$, where $x'^r \equiv \frac{dx^r}{ds}$, show that the equations of the straight line can be written in the form

$$\frac{d}{ds} \left(\frac{\partial \Phi}{\partial x'^r} \right) - \frac{\partial \Phi}{\partial x^r} = 0. \quad (\alpha)$$

Here we are denoting derivatives with respect to s by dashes. We have

$$\frac{\partial \Phi}{\partial x'^r} = \frac{\partial}{\partial x'^r} (g_{mn} x'^m x'^n) = 2 g_{rm} x'^m,$$

Therefore

$$\frac{d}{ds} \left(\frac{\partial \Phi}{\partial x'^r} \right) = 2 g_{rm} x''^m + 2 \frac{\partial g_{rm}}{\partial x^n} x'^m x'^n$$

and

$$\frac{\partial \Phi}{\partial x^r} = \frac{\partial g_{mn}}{\partial x^r} x'^m x'^n.$$

Consequently

$$\begin{aligned} \frac{d}{ds} \left(\frac{\partial \Phi}{\partial x'^r} \right) - \frac{\partial \Phi}{\partial x^r} &= 2 g_{rp} x''^p + 2 [m n, r] x'^m x'^n \\ &= 2 g_{rp} \left(x''^p + \left\{ \begin{matrix} p \\ mn \end{matrix} \right\} x'^m x'^n \right). \end{aligned}$$

But the equations of the straight line are

$$x''^p + \left\{ \begin{matrix} p \\ mn \end{matrix} \right\} x'^m x'^n = 0.$$

Hence our theorem is proved.

This result is sometimes of value in calculating the symbols $\left\{ \begin{matrix} r \\ mn \end{matrix} \right\}$. If we form the equations (α) and solve them for x''^r , we see that on the other side of the equation we have $-\left\{ \begin{matrix} r \\ mn \end{matrix} \right\} x'^m x'^n$. In other words, the symbols we require are the coefficients of the quadratic form in x'^r .

2. Use the method explained in Ex. 1 to find the Christoffel symbols for cylindrical polar coordinates and also for spherical polar coordinates.

EXAMPLES XIII.

1. Prove that $\frac{\delta^2 \lambda^r}{\delta s^2} = \frac{d\kappa}{ds} \mu^r + \kappa (\tau \nu^r - \kappa \lambda^r)$.
2. Show that $\tau = \frac{1}{\kappa^2} \varepsilon_{rst} \lambda^r \frac{\delta \lambda^s}{\delta s} \frac{\delta^2 \lambda^t}{\delta s^2}$.
3. Prove the formulae $\frac{\delta^2 \mu^r}{\delta s^2} = \frac{d\tau}{ds} \nu^r - (\kappa^2 + \tau^2) \mu^r - \frac{d\kappa}{ds} \lambda^r$,
 $\frac{\delta^2 \nu^r}{\delta s^2} = \tau (\kappa \lambda^r - \tau \nu^r) - \frac{d\tau}{ds} \mu^r$.
4. Prove that $\varepsilon_{rst} \frac{\delta \lambda^r}{\delta s} \frac{\delta^2 \lambda^s}{\delta s^2} \frac{\delta^2 \lambda^t}{\delta s^2} = \kappa^5 \frac{d}{ds} \left(\frac{\tau}{\kappa} \right)$,
 $\varepsilon_{rst} \frac{\delta \nu^r}{\delta s} \frac{\delta^2 \nu^s}{\delta s^2} \frac{\delta^2 \nu^t}{\delta s^2} = \tau^5 \frac{d}{ds} \left(\frac{\kappa}{\tau} \right)$.
5. Find the curvature and torsion at any point of the curve $x^1 = a$, $x^2 = t$, $x^3 = ct$ where $ds^2 = (dx^1)^2 + (x^1 dx^2)^2 + (dx^3)^2$.
 [This is the circular helix in cylindrical polar coordinates.]
6. Prove that the tangent vector in Ex. 5 always makes a constant angle with the vector $(0, 0, 1)$ at each point.
7. If $(a \lambda^r + b \mu^r + c \nu^r)$ forms a parallel vector-field along C , prove that $\frac{da}{ds} - \kappa b = 0$, $\frac{db}{ds} + \kappa a - \tau c = 0$, $\frac{dc}{ds} + \tau b = 0$.
8. If small distances, s , are taken along a curve and the tangent straight line at P giving the points Q and R , prove that

$$\left(\frac{\overrightarrow{QR}}{s} \right) \rightarrow \kappa \mu^r.$$

9. If two curves C, C' have the same tangent at P , and Q, R are two points taken at small distances s along the curves, prove that

$$\left(\frac{\overrightarrow{QR}}{s} \right) \rightarrow \kappa \mu^r - \kappa' \mu'^r, \quad \left(\frac{QR}{s} \right)^2 = \kappa^2 + \kappa'^2 - 2\kappa\kappa' \cos \theta,$$

where θ is the angle between their principal normals.

10. Let a curve C be drawn from a point P to a point Q and let \bar{C} be a slightly varied curve whose coordinates at any point differ from those of the corresponding point on C by η^r , an infinitesimal vector, prove that the difference in the lengths of the curves is approximately

$$\delta L = [\lambda_r \eta^r]_P^Q - \int_P^Q \kappa \mu_r \eta^r ds.$$

11. Deduce from Ex. 10 that if C is the curve of stationary length for all curves passing through P and Q , then $\kappa = 0$ and C is a straight line.

CHAPTER XIV

INTRINSIC GEOMETRY OF A SURFACE

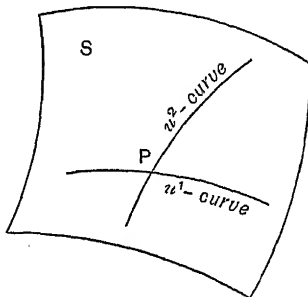
1. Curvilinear coordinates on a surface.

Let (y^1, y^2, y^3) , as before, be the Cartesian coordinates of a point in space. A *surface* is defined in general as the locus of a point whose coordinates are functions of two independent parameters. Thus the equations of a surface are of the form

$$y^r = y^r(u^1, u^2) \quad (r = 1, 2, 3), \quad (1)$$

when u^1 and u^2 are the parameters. In other words, any point on the surface is uniquely determined by two numbers u^1, u^2 and we can therefore call these quantities *coordinates of a point on the surface*. It must be remembered that again 1 and 2 are indices distinguishing the coordinates and u^2 does *not* mean u squared.

Let us examine the geometrical significance of these coordinates (u^1, u^2) . If u^2 is kept constant and u^1 alone varies, then the point (1) depends upon a single parameter and therefore describes a curve. Moreover this curve lies wholly on the surface and we see that when we give u^2 a series of constant values we obtain a family of curves on the surface. These curves we call the u^1 -curves and along them u^1 alone varies, that is, the equations of these curves are $u^2 = \text{const.}$ Hence if we are told that the point P has a certain coordinate u^2_0 , this means that P lies on a certain u^1 -curve, namely $u^2 = u^2_0$.



In the same way we have another family of curves, given by $u^1 = \text{const.}$, along which u^2 alone varies. These are called the u^2 -curves and we now see that each point of the surface is determined as the intersection of a member of each family. Thus geometrically the two coordinates (u^1, u^2) give the two curves, one of each family, which pass through the point. We shall often refer to the u^1 - and u^2 -curves briefly as *the coordinate curves* and we shall call u^1, u^2 a system of *curvilinear coordinates on the surface*.

All the properties of a surface which can be described without referring to the surrounding space are called *intrinsic properties of the surface* and their description constitutes the *intrinsic geometry of the surface*. It is this intrinsic geometry we wish to deal with in the present chapter and we shall find that the curvilinear coordinates (u^1, u^2) are the most appropriate coordinates to use for this purpose.

There is obviously an infinite number of possible coordinate systems by means of which we may determine points on our surface. We may, in fact, take as coordinate curves any two families of curves which fulfil the condition that each member of one family intersects each member of the other in one and only one point. If \bar{u}^1, \bar{u}^2 is another coordinate system on the surface, then \bar{u}^1, \bar{u}^2 are clearly functions of u^1, u^2 and conversely. That is, there is a functional transformation between the coordinates of the form

$$\bar{u}^1 = f^1(u^1, u^2), \quad \bar{u}^2 = f^2(u^1, u^2) \quad (2)$$

and this transformation is reversible, so that

$$u^1 = \varphi^1(\bar{u}^1, \bar{u}^2), \quad u^2 = \varphi^2(\bar{u}^1, \bar{u}^2). \quad (3)$$

EXAMPLES.

1. Show that the surface $y^3 = f(y^1, y^2)$ may be put into the parametric form $y^1 = u^1$, $y^2 = u^2$, $y^3 = f(u^1, u^2)$. What are the coordinate curves?

2. Show that $y^1 = a \cos u^1 \cos u^2$, $y^2 = a \cos u^1 \sin u^2$, $y^3 = a \sin u^1$ are the parametric equations of a sphere and determine the coordinate curves.

3. Show that

$$\frac{y^1}{a} = \frac{u^1 + u^2}{1 + u^1 u^2}, \quad \frac{y^2}{b} = \frac{1 - u^1 u^2}{1 + u^1 u^2}, \quad \frac{y^3}{c} = \frac{u^1 - u^2}{1 + u^1 u^2}.$$

are the equations of a hyperboloid of one sheet and that the coordinate curves are the generators.

2. The conventions regarding Greek indices. Surface tensors.

We therefore see that the curvilinear coordinates on a surface are *two* in number and we are now dealing with *two* variables (u^1, u^2) . Up to the present we have used *italic indices* to denote our variables, which have been three in number, and we have adopted certain conventions regarding these italic indices,

namely, that such an index takes the range of values 1 to 3 and a dummy index is to be summed from 1 to 3. We shall find it convenient to adopt an indicial notation for our new variables (u^1, u^2) also, but to avoid changing any conventions we have made for italic indices we shall now adopt *Greek indices*. Thus we may write our variables

$$u^\alpha (\alpha = 1, 2). \quad \dots \dots \dots (4)$$

We shall also make similar conventions regarding these Greek indices and they will be two in number.

A repeated or dummy Greek index in any term is to be summed from 1 to 2.

A free or unrepeated Greek index is to have the range of values 1, 2.

It will be observed that all the remarks of Chapter I can now be repeated with slight modifications when we use *Greek indices* and the new conventions respecting them. For example, a system of the second order will be denoted by $a_{\alpha\beta}$ and will stand for any one of the array of quantities

$$\begin{array}{cc} a_{11}, & a_{12} \\ a_{21}, & a_{22}. \end{array}$$

Again we shall have

$$a_{\alpha\beta} u^\alpha u^\beta = \sum_{\alpha, \beta=1}^2 a_{\alpha\beta} u^\alpha u^\beta \equiv a_{11} (u^1)^2 + a_{12} u^1 u^2 + a_{21} u^2 u^1 + a_{22} (u^2)^2.$$

The corresponding skew-symmetric *e-systems* in Greek indices are *double systems*, namely, $e_{\alpha\beta}$ and $e^{\alpha\beta}$, which have the following values

$$\begin{array}{ll} = e_{22} = 0, & = -e_{21} = 1, \\ e_{11} - e_{22} = 0, & = -e^{21} = 1, \end{array} \quad (5)$$

and we have now two Kronecker deltas, defined by

$$\delta_{\lambda\mu}^{\alpha\beta} = e^{\alpha\beta} e_{\lambda\mu}, \quad \delta_\lambda^\alpha = \delta_{\lambda\beta}^{\alpha\beta}. \quad \dots \dots \dots (6)$$

The first has the value 0 unless α, β and λ, μ are permutations of 1, 2, in which case it is +1 if the two permutations are the same and -1 if they are opposite. The second delta has the usual properties that it equals 1 if α and λ are equal and is zero if they are unequal. We leave to the student the development of the theory of determinants of the second order analogous to that given in Chapter I.

The transformations (2) and (3) between two different systems of curvilinear coordinates on a surface can be written in the brief form

$$\begin{array}{l} \bar{u}^\alpha = f^\alpha (u^1, u^2), \\ u^\alpha = \varphi^\alpha (\bar{u}^1, \bar{u}^2). \end{array} \quad \dots \dots \dots (7)$$

We have therefore a theory of tensors of different orders for the transformations in these variables which is the same in every respect as that already developed for the variables x' . Thus a triple system $a_{\beta\gamma}^{\alpha}$ in Greek indices is a *relative tensor of weight M , contravariant in α and covariant in β and γ* , when its components $\bar{a}_{\sigma\tau}^{\rho}$ in the new variables satisfy the relations

$$\bar{a}_{\sigma\tau}^{\rho} = \left| \frac{\partial u^{\mu}}{\partial \bar{u}^{\nu}} \right|^M \frac{\partial \bar{u}^{\rho}}{\partial u^{\alpha}} \frac{\partial u^{\beta}}{\partial \bar{u}^{\sigma}} \frac{\partial u^{\gamma}}{\partial \bar{u}^{\tau}} a_{\beta\gamma}^{\alpha} \dots \quad (8)$$

The theorems in tensor algebra proved in Chapter II (p. 19) are of course equally true here as are also the observations regarding tensor-fields at p. 133. When $M = 0$ in (8) the tensor $a_{\beta\gamma}^{\alpha}$ is called as before an *absolute tensor* and this is the most important class of tensors for us.

If we want to distinguish these tensors in *Greek* indices from those in *italic* indices we may call them *surface tensors* since they are tensors under a transformation of surface coordinates. The others we may call *space tensors* since they are tensors under a transformation of space coordinates.

EXAMPLES.

1. Show that the determinant $a \equiv |a_{\alpha\beta}|$ satisfies the equations

$$e^{\gamma\delta} a_{\gamma\alpha} a_{\delta\beta} = e^{\gamma\delta} a_{\alpha\gamma} a_{\beta\delta} = a e_{\alpha\beta}.$$

2. Deduce from Ex. 1 that

$$2! a = e^{\alpha\gamma} e^{\beta\delta} a_{\alpha\beta} a_{\gamma\delta}.$$

3. Show that the cofactor of $a_{\alpha\beta}$ in a is $A^{\alpha\beta}$ where

$$A^{\alpha\beta} = e^{\alpha\gamma} e^{\beta\delta} a_{\gamma\delta}.$$

4. Prove that

$$e_{\alpha\beta} = \left| \frac{\partial w^{\mu}}{\partial \bar{u}^{\nu}} \right|^{-1} \frac{\partial w^{\rho}}{\partial \bar{u}^{\alpha}} \frac{\partial w^{\sigma}}{\partial \bar{u}^{\beta}} e_{\rho\sigma}, \quad e^{\alpha\beta} = \left| \frac{\partial w^{\mu}}{\partial \bar{u}^{\nu}} \right| \frac{\partial \bar{u}^{\alpha}}{\partial w^{\rho}} \frac{\partial \bar{u}^{\beta}}{\partial w^{\sigma}} e^{\rho\sigma}$$

and deduce that $e_{\alpha\beta}$, $e^{\alpha\beta}$ are relative tensors of weights -1 and 1 respectively.

5. If $a_{\alpha\beta}$ is an absolute tensor, deduce from Exs. 2, 3, 4 that a is a relative invariant and $A^{\alpha\beta}$ a relative tensor both of weight 2, and hence that the cofactor $A^{\alpha\beta}$, divided by a , is an absolute tensor.

3. The element of length and the metric tensor.

Let P be a point on the surface whose coordinates are u^α and let Q be a neighbouring point of the surface with coordinates $u^\alpha + du^\alpha$. We shall denote by y^r and $y^r + dy^r$ the Cartesian coordinates of P and Q in space. From the relations (1), p. 163, we have

$$dy^r = \frac{\partial y^r}{\partial u^\alpha} du^\alpha. \quad \dots \dots \dots (9)$$

If we let ds be the elementary distance between P and Q we know that

$$ds^2 = \sum_1^3 (dy^r)^2.$$

Hence we conclude from (9) that

$$ds^2 = a_{\alpha\beta} du^\alpha du^\beta, \quad \dots \dots \dots (10)$$

where we have put

$$a_{\alpha\beta} = \frac{\partial y^r}{\partial u^\alpha} \frac{\partial y^r}{\partial u^\beta} = \sum_{r=1}^3 \frac{\partial y^r}{\partial u^\alpha} \frac{\partial y^r}{\partial u^\beta}. \quad \dots \dots \dots (11)$$

We see that $a_{\alpha\beta}$ is a function of the u 's and is *symmetric* in α, β . Also since ds is an invariant it follows from (10) that $a_{\alpha\beta} du^\alpha du^\beta$ is an invariant for arbitrary values of the contravariant vector du^α . Consequently our quotient law tells us that $a_{\alpha\beta}$ is a *double covariant tensor*, which we call the *fundamental* or *metric tensor*, and the element of length on the surface is given by the formula (10).

If we denote by a the determinant $|a_{\alpha\beta}|$ and by $a^{\alpha\beta}$ the cofactor of $a_{\alpha\beta}$ in a , divided by a , we prove as at p. 135 that $a^{\alpha\beta}$ is a *contravariant tensor*, and we have

$$a^{\alpha\beta} a_{\beta\gamma} = \delta^\alpha_\gamma. \quad \dots \dots \dots (12)$$

Also if we form the systems

$$\varepsilon_{\alpha\beta} = \sqrt{a} e_{\alpha\beta}, \quad \varepsilon^{\alpha\beta} = \frac{1}{\sqrt{a}} e^{\alpha\beta} \quad \dots \dots \dots (13)$$

we see that these are also tensors which we may call the ε -systems.

When we are given a contravariant vector A^α we define the *magnitude* A of A^α by the equation

$$A = (a_{\alpha\beta} A^\alpha A^\beta)^{\frac{1}{2}}, \quad \dots \dots \dots (14)$$

and similarly the magnitude of a covariant vector B_α is

$$B = (a^{\alpha\beta} B_\alpha B_\beta)^{\frac{1}{2}}. \quad \dots \quad (15)$$

Also a *unit* vector is one whose magnitude is unity and hence λ^α will be a unit vector if it satisfies the equation

$$a_{\alpha\beta} \lambda^\alpha \lambda^\beta = 1. \quad \dots \quad (16)$$

If we divide equation (10) by ds^2 we have

$$a_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = 1, \quad \dots \quad (17)$$

that is, the vector $\frac{du^\alpha}{ds}$ is a unit vector.

With the aid of the tensors $a_{\alpha\beta}$, $a^{\alpha\beta}$ we can raise and lower indices in the usual manner and these operations obey the same laws as in the case of space tensors.

EXAMPLES.

Show that the ds^2 for the following surfaces have the values mentioned.

1. $y^1 = u^1, y^2 = u^2, y^3 = f(u^1, u^2); ds^2 = (1 + f_1^2) (du^1)^2 + 2 f_1 f_2 du^1 du^2 + (1 + f_2^2) (du^2)^2$, where $f_\alpha = \frac{\partial f}{\partial u^\alpha}$.

2. *The sphere.* $y^1 = a \cos u^1 \cos u^2, y^2 = a \cos u^1 \sin u^2, y^3 = a \sin u^1; ds^2 = a^2 (du^1)^2 + a^2 (\cos u^1)^2 (du^2)^2$.

3. *The circular cylinder.* $y^1 = a \cos u^1, y^2 = a \sin u^1, y^3 = u^2; ds^2 = a^2 (du^1)^2 + (du^2)^2$.

4. $y^1 = u^1 \cos u^2, y^2 = u^1 \sin u^2, y^3 = 0; ds^2 = (du^1)^2 + (u^1 du^2)^2$.

[This surface is the plane $y^3 = 0$ and the coordinates are polar coordinates.]

4. Directions on a surface. Angle between two directions.

If we divide (9) by ds we get

$$\frac{dy^r}{ds} = \frac{\partial y^r}{\partial u^\alpha} \frac{du^\alpha}{ds}. \quad \dots \quad (18)$$

Now we know that the direction PQ is given completely by $\frac{dy^r}{ds}$ in Cartesian coordinates and (18) shows that $\frac{dy^r}{ds}$ is given when $\frac{du^\alpha}{ds}$ is given. Hence the direction PQ on the surface is

completely determined by the unit vector $\frac{du^\alpha}{ds}$. But if we are given any unit contravariant vector λ^α we can always find $\frac{du^\alpha}{ds}$ such that

$$\frac{du^\alpha}{ds} = \lambda^\alpha. \quad (19)$$

and it follows that a unit surface vector λ^α determines a unique direction on the surface. Also any contravariant vector A^α consists of an invariant A , its magnitude, and a unit vector A^α/A . Consequently every contravariant surface vector defines a magnitude and a direction on the surface.

Let us next consider two directions on the surface at P , defined by the vectors $\frac{du^\alpha}{ds}, \frac{\delta u^\alpha}{\delta s}$. The corresponding space vectors in the Cartesian coordinates are given by

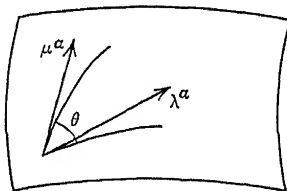
$$\frac{dy^r}{ds} = \frac{\partial y^r}{\partial u^\alpha} \frac{du^\alpha}{ds}, \quad \frac{\delta y^r}{\delta s} = \frac{\partial y^r}{\partial u^\beta} \frac{\delta u^\beta}{\delta s}, \quad . . . (20)$$

and we know that the angle θ between the two directions is given by

$$\cos \theta = \frac{dy^r}{ds} \frac{\delta y^r}{\delta s} = \sum_{r=1}^3 \frac{dy^r}{ds} \frac{\delta y^r}{\delta s}.$$

Hence substituting for $\frac{dy^r}{ds}$ and $\frac{\delta y^r}{\delta s}$ their values in (20) and remembering the values of $a_{\alpha\beta}$, we obtain

$$\cos \theta = a_{\alpha\beta} \frac{du^\alpha}{ds} \frac{\delta u^\beta}{\delta s}. \quad . (21)$$



If $\lambda^\alpha, \mu^\alpha$ are two unit vectors the two directions defined by them are given by

$$\frac{du^\alpha}{ds} = \lambda^\alpha, \quad \frac{\delta u^\alpha}{\delta s} = \mu^\alpha,$$

and we conclude that the angle θ between the two directions $\lambda^\alpha, \mu^\alpha$ is given by

$$\cos \theta = a_{\alpha\beta} \lambda^\alpha \mu^\beta, \quad (22)$$

These directions will be orthogonal if $\theta = \pi/2$ and hence the condition that the directions $\lambda^\alpha, \mu^\alpha$ be orthogonal is

$$a_{\alpha\beta} \lambda^\alpha \mu^\beta = 0. \quad (23)$$

If we take the direction v^α defined by

$$v^\alpha = \varepsilon^{\beta\alpha} \lambda_\beta, \quad \dots \dots \dots (24)$$

where λ_α is the associated vector of the unit vector λ^α , we see that

$$a_{\alpha\gamma} v^\alpha v^\gamma = a_{\alpha\gamma} \varepsilon^{\beta\alpha} \varepsilon^{\delta\gamma} \lambda_\beta \lambda_\delta.$$

But

$$a_{\alpha\gamma} \varepsilon^{\beta\alpha} \varepsilon^{\delta\gamma} = \frac{1}{a} a_{\alpha\gamma} e^{\beta\alpha} e^{\delta\gamma} = a^{\beta\delta}.$$

Therefore

$$a_{\alpha\gamma} v^\alpha v^\gamma = a^{\beta\delta} \lambda_\beta \lambda_\delta = a_{\beta\delta} \lambda^\beta \lambda^\delta = 1,$$

which shows that (24) is a unit vector. Moreover

$$a_{\alpha\gamma} v^\alpha \lambda^\gamma = a_{\alpha\gamma} \varepsilon^{\beta\alpha} \lambda_\beta \lambda^\gamma = \varepsilon^{\beta\alpha} \lambda_\beta \lambda_\alpha = 0$$

and we conclude that v^α is the unit vector orthogonal to λ^α . If we multiply (24) by v_α and sum α from 1 to 2, then

$$1 = \varepsilon^{\beta\alpha} \lambda_\beta v_\alpha = \varepsilon_{\alpha\beta} \lambda^\alpha v^\beta,$$

and we see that $\varepsilon_{\alpha\beta} \lambda^\alpha v^\beta$ is positive. Now

$$\varepsilon_{\alpha\beta} \lambda^\alpha \mu^\beta = -\varepsilon_{\alpha\beta} \mu^\alpha \lambda^\beta, \quad \dots \dots \dots (25)$$

where $\lambda^\alpha, \mu^\alpha$ are any two vectors, and we say that the rotation from λ^α to μ^α is positive if $\varepsilon_{\alpha\beta} \lambda^\alpha \mu^\beta$ is positive. Consequently the vector (24) is such that the rotation $\lambda^\alpha \rightarrow v^\alpha$ is positive.

EXAMPLES.

1. If unit vectors $\varepsilon_{(1)}^\alpha, \varepsilon_{(2)}^\alpha$ are taken in the directions of the coordinate curves, show that

$$\varepsilon_{(1)}^\alpha = \frac{1}{\sqrt{a_{11}}} \delta_1^\alpha, \quad \varepsilon_{(2)}^\alpha = \frac{1}{\sqrt{a_{22}}} \delta_2^\alpha.$$

Along the coordinate curve $u^2 = \text{const}$, we have $du^2 = 0$, and thus the element of distance along it is given by $ds^2 = a_{11}(du^1)^2$. Consequently the unit vector $\varepsilon_{(1)}^\alpha$ is

$$\varepsilon_{(1)}^\alpha = \left(\frac{du^1}{ds}, \frac{du^2}{ds} \right) = \left(\frac{1}{\sqrt{a_{11}}}, 0 \right) = \frac{1}{\sqrt{a_{11}}} \delta_1^\alpha$$

and similarly for the vector $\varepsilon_{(2)}^\alpha$.

2. If $\lambda^\alpha, \mu^\alpha$ are two unit vectors such that the rotation $\lambda^\alpha, \mu^\alpha$ is positive, show that $\varepsilon_{\alpha\beta} \lambda^\alpha \mu^\beta = \sin \theta$.

3. If ω is the angle between the coordinate curves, show that $\cos \omega = \frac{a_{12}}{\sqrt{a_{11}a_{22}}}$. Consequently the condition that the coordinate curves be orthogonal is $a_{12} = 0$, and the coordinates are then said to be orthogonal curvilinear coordinates.

4. Show that the rotation from the u^1 -curve to the u^2 -curve is positive and $\sin \omega = \frac{\sqrt{a}}{\sqrt{a_{11}a_{22}}}$.

5. *Element of area.* Prove that the element of area dS on the surface is given by $dS = \sqrt{a} du^1 du^2$.

[If we take small distances ds_1, ds_2 along the coordinate curves, then $dS = ds_1 ds_2 \sin \omega$, and we have $ds_1 = \sqrt{a_{11}} du^1, ds_2 = \sqrt{a_{22}} du^2$.]

5. The equations of a geodesic.

Any curve on the surface is determined by giving the coordinates u^α in terms of a single parameter t . Thus the equations of a curve will be of the form

$$u^\alpha = f^\alpha(t). \quad (26)$$

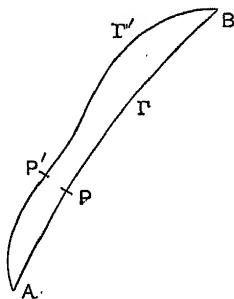
Also if we denote by dots differentiations with respect to t , we see that the length of the curve between the points A and B on it is

$$L = \int_A^B (\alpha_{\alpha\beta} \dot{u}^\alpha \dot{u}^\beta)^{\frac{1}{2}} dt. \quad (27)$$

Let us consider curves passing through two fixed points A and B . Of all such curves there is in general one and only one whose length from A to B is less than that of the others. This curve is called *the geodesic joining A and B* , and if our surface were a plane we know that the geodesic would be the straight line. Our present problem is to find the equations of this geodesic through A and B .

Let us denote by Γ the geodesic through A and B and let us take a neighbouring curve Γ' also passing through A and B . We set up a one-to-one correspondence between the points of Γ and Γ' in such a way that if P , coordinates u^α , and P' , coordinates u'^α , are two such corresponding points, then PP' is a small vector. It may therefore be written $\varepsilon \omega^\alpha$, where ε is an infinitesimal and ω^α is a finite contravariant vector. Hence

$$u'^\alpha = u^\alpha + \varepsilon \omega^\alpha, \quad (28)$$



and as we move along Γ we see that ω^r is a function of the parameter t which vanishes at the points A and B . Thus L' , the length of Γ' between A and B , is a function of ε and can be expanded in ascending powers of ε by means of Taylor's series. Remarking that for $\varepsilon = 0$ we are on Γ , we obtain

$$L' = L + \varepsilon \left(\frac{\partial L'}{\partial \varepsilon} \right)_0 + \frac{1}{2!} \varepsilon^2 \left(\frac{\partial^2 L'}{\partial \varepsilon^2} \right)_0 + \dots, \quad (29)$$

where the suffix 0 means that ε has been put equal to zero after differentiation. The second term on the right of (29) is called *the first variation of the length L* and is usually denoted by δL . Since Γ is a geodesic its length is less than that of any other curve Γ' through A and B . In other words, L' is a minimum for $\varepsilon = 0$, and consequently, by the ordinary rules of the differential calculus, $\left(\frac{\partial L'}{\partial \varepsilon} \right)_0$ must vanish, that is, *the first variation δL is zero if Γ is a geodesic*. We proceed to find an explicit expression for δL , and by expressing that it vanishes for all neighbouring curves we shall arrive at the equations of Γ .

Let us put

$$\varphi(u, \dot{u}) = (a_{\alpha\beta} \dot{u}^\alpha \dot{u}^\beta)^{\frac{1}{2}} \quad (30)$$

where u, \dot{u} stand symbolically for u^1, u^2 and \dot{u}^1, \dot{u}^2 respectively. Thus

$$\varphi' \equiv \varphi(u', \dot{u}') \equiv \varphi(u + \varepsilon \omega, \dot{u} + \varepsilon \dot{\omega}),$$

and

$$\left(\frac{\partial \varphi'}{\partial \varepsilon} \right)_0 = \frac{\partial \varphi}{\partial u^\alpha} \omega^\alpha + \frac{\partial \varphi}{\partial \dot{u}^\alpha} \dot{\omega}^\alpha.$$

But

$$L' = \int_A^B \varphi' dt.$$

Hence

$$\delta L = \varepsilon \left(\frac{\partial L'}{\partial \varepsilon} \right)_0 = \varepsilon \int_A^B \left\{ \frac{\partial \varphi}{\partial u^\alpha} \omega^\alpha + \frac{\partial \varphi}{\partial \dot{u}^\alpha} \dot{\omega}^\alpha \right\} dt.$$

When we integrate by parts we have

$$\int_A^B \frac{\partial \varphi}{\partial \dot{u}^\alpha} \dot{\omega}^\alpha dt = \left[\frac{\partial \varphi}{\partial \dot{u}^\alpha} \omega^\alpha \right]_A^B - \int_A^B \frac{d}{dt} \left(\frac{\partial \varphi}{\partial \dot{u}^\alpha} \right) \omega^\alpha dt,$$

and the integrated part disappears since ω^α is zero at A and B .

Thus

$$\delta L = \varepsilon \int_A^B \left\{ \frac{\partial \varphi}{\partial u^\alpha} - \frac{d}{dt} \left(\frac{\partial \varphi}{\partial \dot{u}^\alpha} \right) \right\} \omega^\alpha dt, \dots \quad (31)$$

and we have found an expression for the first variation of L .

If Γ is a geodesic δL must be zero for all neighbouring curves through A and B , that is, (31) must vanish for all arbitrary values of the vector ω^α along Γ . Therefore we must have

$$\frac{d}{dt} \left(\frac{\partial \varphi}{\partial \dot{u}^\alpha} \right) - \frac{\partial \varphi}{\partial u^\alpha} = 0. \dots \quad (32)$$

These differential equations, together with the conditions that Γ passes through the given points A and B , will in general define the curve Γ completely. They are therefore the equations of Γ , and we shall write them in a more explicit form.

We have

$$\frac{\partial \varphi}{\partial \dot{u}^\alpha} = \frac{a_{\alpha\beta} \dot{u}^\beta}{\varphi}, \quad \frac{\partial \varphi}{\partial u^\alpha} = \frac{1}{2\varphi} \frac{\partial a_{\beta\gamma}}{\partial u^\alpha} \dot{u}^\beta \dot{u}^\gamma.$$

and (32) reads

$$\frac{d}{dt} \left(\frac{a_{\alpha\beta} \dot{u}^\beta}{\varphi} \right) - \frac{1}{2\varphi} \frac{\partial a_{\beta\gamma}}{\partial u^\alpha} \dot{u}^\beta \dot{u}^\gamma = 0.$$

Thus far our parameter along Γ has been perfectly general. We shall now simplify matters if we take as parameter the arc s of the geodesic, in which case (30) gives $\varphi = 1$ along Γ . The equations of Γ become

$$\frac{d}{ds} \left(a_{\alpha\beta} \frac{du^\beta}{ds} \right) - \frac{1}{2} \frac{\partial a_{\beta\gamma}}{\partial u^\alpha} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} = 0,$$

or

$$a_{\alpha\beta} \frac{d^2 u^\beta}{ds^2} + \frac{\partial a_{\alpha\beta}}{\partial u^\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} - \frac{1}{2} \frac{\partial a_{\beta\gamma}}{\partial u^\alpha} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} = 0.$$

If we introduce the symbols

$$[\alpha\beta, \gamma] = \frac{1}{2} \left(\frac{\partial a_{\beta\gamma}}{\partial u^\alpha} + \frac{\partial a_{\alpha\gamma}}{\partial u^\beta} - \frac{\partial a_{\alpha\beta}}{\partial u^\gamma} \right), \dots \quad (33)$$

these equations can be written

$$a_{\alpha\beta} \frac{d^2 u^\beta}{ds^2} + [\beta\gamma, \alpha] \frac{du^\beta}{ds} \frac{du^\gamma}{ds} = 0.$$

Raising the index α , we obtain the final form

$$\frac{d^2 u^\alpha}{ds^2} + \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} = 0, \quad \dots \quad (34)$$

where we have put

$$\left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} = a^{\alpha\delta} [\beta\gamma, \delta]. \quad \dots \quad (35)$$

The quantities (33) and (35) are called the Christoffel symbols of the surface, and reference to p. 141 will show that they are formed from the $a_{\alpha\beta}$ in exactly the same way as the Christoffel symbols of space were formed from the g_{rs} . Moreover we know that the curve of shortest length, or the geodesic, between two points *in space* is the straight line and we see from equations (13), p. 161, that the equations of the geodesic on the surface and of that in space are of exactly the same form.

EXAMPLES.

1. Show that if we put $\psi = a_{\alpha\beta} u'^\alpha u'^\beta$, where $u'^\alpha = \frac{du^\alpha}{ds}$, the equations of the geodesic can be written

$$\frac{d}{ds} \left(\frac{\partial \psi}{\partial u'^\alpha} \right) - \frac{\partial \psi}{\partial u^\alpha} = 0.$$

[These equations are often useful in calculating the Christoffel symbols for any particular coordinate system. For when these equations are solved with respect to the second derivatives of u^α with respect to s , we see from (34) that on the right-hand side of the equation the coefficient of

$$\frac{du^\beta}{ds} \frac{du^\gamma}{ds} \text{ is } - \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\}$$

2. Show that $[\alpha\beta, \gamma]$ and $\left\{ \begin{matrix} \gamma \\ \alpha \beta \end{matrix} \right\}$ are symmetric in α, β , and show that

$$\frac{\partial a_{\alpha\beta}}{\partial u^\gamma} = [\alpha\gamma, \beta] + [\beta\gamma, \alpha], \quad [\alpha\beta, \gamma] = a_{\gamma\delta} \left\{ \begin{matrix} \delta \\ \alpha \beta \end{matrix} \right\}.$$

3. For the surface of Ex. 1, p. 168, show that the Christoffel symbols are

$$\begin{aligned} \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} &= k f_1 f_{11}, & \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} &= k f_1 f_{12}, & \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} &= k f_1 f_{22}, & \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} &= k f_2 f_{11}, \\ \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} &= k f_2 f_{12}, & \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} &= k f_2 f_{22}, \end{aligned}$$

where we have put

$$f_{\alpha} = \frac{\partial f}{\partial u^{\alpha}} \quad f_{\alpha\beta} = \frac{\partial^2 f}{\partial u^{\alpha} \partial u^{\beta}}, \quad \text{and} \quad k = \frac{1}{1 + f_1^2 + f_2^2}.$$

4. *Orthogonal curvilinear coordinates.* If the coordinate curves are orthogonal, prove that

$$(i) \quad a_{12} = a^{12} = 0, \quad a^{11} = \frac{1}{a_{11}}, \quad a_{22} = \frac{1}{a_{22}},$$

$$(ii) \quad [11, 1] = \frac{1}{2} \frac{\partial a_{11}}{\partial u^1}, \quad [12, 1] = \frac{1}{2} \frac{\partial a_{11}}{\partial u^2} = -[11, 2],$$

$$[22, 1] = -\frac{1}{2} \frac{\partial a_{22}}{\partial u^1} = -[12, 2], \quad [22, 2] = \frac{1}{2} \frac{\partial a_{22}}{\partial u^2}.$$

$$(iii) \quad \begin{matrix} 11 & \frac{1}{2a_{11}} \frac{\partial a_{11}}{\partial u^1}, & \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} & \frac{1}{2a_{11}} \frac{\partial a}{\partial u^2}, & -\frac{1}{2a_{11}} \frac{\partial a_{\alpha}}{\partial u^1}, \\ \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} & = -\frac{1}{2a_{22}} \frac{\partial a_{11}}{\partial u^2}, & \frac{1}{2a_{22}} \frac{\partial a_{22}}{\partial u^1}, & \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} & = \frac{1}{2a_{22}} \frac{\partial a_{22}}{\partial u^2}. \end{matrix}$$

5. Obtain the Christoffel symbols for the surfaces of Exs. 2, 3, 4 (p. 168).

6. *The transformation of the Christoffel symbols. Geodesic coordinates.*

We shall now take two coordinate systems u^{α} , \bar{u}^{α} and examine how the Christoffel symbols of the two systems are related. We denote by bars quantities which refer to the \bar{u} -coordinates.

Let us take any geodesic Γ and let u^{α} , \bar{u}^{α} be the coordinates of a point on it referred to the two coordinate systems. These are functions of the arc s of Γ and of course s is an invariant. We have

$$\frac{d\bar{u}^{\alpha}}{ds} = \frac{\partial \bar{u}^{\alpha}}{\partial u^{\rho}} \frac{du^{\rho}}{ds}, \quad \dots \dots \dots (36)$$

and, on differentiation,

$$\frac{d^2 \bar{u}^{\alpha}}{ds^2} = \frac{\partial \bar{u}^{\alpha}}{\partial u^{\rho}} \frac{d^2 u^{\rho}}{ds^2} + \frac{\partial^2 \bar{u}^{\alpha}}{\partial u^{\rho} \partial u^{\sigma}} \frac{du^{\rho}}{ds} \frac{du^{\sigma}}{ds}. \quad \dots \dots (37)$$

But the point lies on the geodesic and we must have in both coordinate systems

$$\begin{matrix} \frac{d^2 \bar{u}^{\alpha}}{ds^2} & \overline{\alpha} \\ & \beta \gamma \end{matrix} \left\{ \frac{d\bar{u}^{\beta}}{ds} \frac{d\bar{u}^{\gamma}}{ds} \right.$$

$$\frac{d^2 u^{\rho}}{ds^2} \quad \quad \rho \quad \left\{ \frac{du^{\sigma}}{ds} \frac{du^{\tau}}{ds} \right.$$

$$\sigma \tau$$

Therefore

$$\left\{ \begin{array}{c} \alpha \\ \beta \gamma \end{array} \right\} \frac{d\bar{u}^\beta}{ds} \frac{d\bar{u}^\gamma}{ds} = \frac{\partial \bar{u}^\alpha}{\partial \bar{u}^\varrho} \left\{ \begin{array}{c} \varrho \\ \sigma \tau \end{array} \right\} \frac{du^\sigma}{ds} \frac{du^\tau}{ds} - \frac{\partial^2 \bar{u}^\alpha}{\partial u^\sigma \partial u^\tau} \frac{du^\sigma}{ds} \frac{du^\tau}{ds},$$

or, from (36),

$$\left(\frac{\partial^2 \bar{u}^\alpha}{\partial u^\sigma \partial u^\tau} - \left\{ \begin{array}{c} \varrho \\ \sigma \tau \end{array} \right\} \frac{\partial \bar{u}^\alpha}{\partial \bar{u}^\varrho} + \left\{ \begin{array}{c} \alpha \\ \beta \gamma \end{array} \right\} \frac{\partial \bar{u}^\beta}{\partial u^\sigma} \frac{\partial \bar{u}^\gamma}{\partial u^\tau} \right) \frac{du^\sigma}{ds} \frac{du^\tau}{ds} = 0. \quad (38)$$

The geodesic Γ was chosen quite arbitrarily and hence (38) is true for all values of $\frac{du^\varrho}{ds}$. Also the coefficient of $\frac{du^\sigma}{ds} \frac{du^\tau}{ds}$ in (38) is easily seen to be symmetric in σ, τ . Consequently

$$\frac{\partial^2 \bar{u}^\alpha}{\partial u^\sigma \partial u^\tau} - \left\{ \begin{array}{c} \varrho \\ \sigma \tau \end{array} \right\} \frac{\partial \bar{u}^\alpha}{\partial \bar{u}^\varrho} + \left\{ \begin{array}{c} \alpha \\ \beta \gamma \end{array} \right\} \frac{\partial \bar{u}^\beta}{\partial u^\sigma} \frac{\partial \bar{u}^\gamma}{\partial u^\tau} = 0, \quad (39)$$

which give us the relations connecting the Christoffel symbols of the two systems. Since it involves the second partial derivatives of \bar{u}^α with respect to u^ϱ , we see that *the Christoffel symbols are not tensors*. Starting from the formulae

$$\frac{du^\alpha}{ds} = \frac{\partial u^\alpha}{\partial \bar{u}^\varrho} \frac{d\bar{u}^\varrho}{ds},$$

we deduce in the same way the reciprocal relations

$$\frac{\partial^2 u^\alpha}{\partial \bar{u}^\sigma \partial \bar{u}^\tau} - \left\{ \begin{array}{c} \varrho \\ \sigma \tau \end{array} \right\} \frac{\partial u^\alpha}{\partial \bar{u}^\varrho} + \left\{ \begin{array}{c} \alpha \\ \beta \gamma \end{array} \right\} \frac{\partial u^\beta}{\partial \bar{u}^\sigma} \frac{\partial u^\gamma}{\partial \bar{u}^\tau} = 0. \quad (40)$$

We have already seen that if we take rectilinear coordinates in space then the fundamental metric tensor of space, g_{rs} , has constant components in such a coordinate system, and the Christoffel symbols are *everywhere* zero. It is not possible in general to find such coordinates on a surface but we shall now show that *we can always choose a coordinate system in which all the Christoffel symbols vanish at a given point O* . Coordinates of this nature are called *geodesic coordinates for the point O* .

Let u^α be a given coordinate system and let u_0^α be the coordinates of the given point O in this system. If coordinates \bar{u}^α exist such that all the Christoffel symbols in it vanish at O , then (40) tells us that we must have *at the point O*

$$\frac{\partial^2 u^\alpha}{\partial \bar{u}^\sigma \partial \bar{u}^\tau} + \left\{ \begin{array}{c} \alpha \\ \beta \gamma \end{array} \right\} \frac{\partial u^\beta}{\partial \bar{u}^\sigma} \frac{\partial u^\gamma}{\partial \bar{u}^\tau} = 0; \quad (41)$$

and, conversely, if these equations are satisfied then $\left\{ \frac{\rho}{\sigma\tau} \right\}$ all vanish at O . Let us make the transformation of variables

$$u^\alpha = u_0^\alpha + \bar{u}^\alpha - \frac{1}{2} \left\{ \frac{\alpha}{\sigma\tau} \right\}_0 \bar{u}^\sigma \bar{u}^\tau. \quad \dots (42)$$

We see that the point O is given in the new coordinates by $\bar{u}^\alpha = 0$ and that

$$\frac{\partial u^\alpha}{\partial \bar{u}^\beta} = \delta_\beta^\alpha$$

at O . Also

$$\frac{\partial^2 u^\alpha}{\partial \bar{u}^\sigma \partial \bar{u}^\tau} = - \left\{ \frac{\alpha}{\sigma\tau} \right\}_0,$$

which shows that (41) are satisfied at the given point. Hence the new variables are geodesic coordinates at O . It is important to notice that in geodesic coordinates the Christoffel symbols are generally not zero everywhere but only at the point in question.

EXAMPLES.

1. *Geodesic polar coordinates. Find the line-element when the coordinate curves are the geodesics passing through a given point O and their orthogonal trajectories.*

Let us take as u^1 -curves the geodesics passing through a given point O and let us take as u^2 the distance along each geodesic measured from O . We may take as u^2 the angle which the geodesic OP makes with a given geodesic OC . We see therefore that the coordinates (u^1, u^2) are exactly analogous to polar coordinates in the plane. Such coordinates may be appropriately called *geodesic polar coordinates*.

Along the geodesic $u^2 = \text{constant}$ we have $du^2 = 0$, $ds = du^1$. Hence $a_{11} = 1$ and

$$\frac{du^1}{ds} = 1, \quad \frac{du^2}{ds} = 0.$$

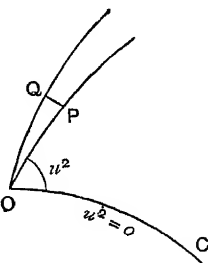
Also these curves must satisfy the equations of geodesics (34), p. 174, that is,

$$\left\{ \begin{smallmatrix} 1 \\ 11 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \right\} = 0.$$

Therefore, since $[11, r] = a_{rs} \left\{ \begin{smallmatrix} s \\ 11 \end{smallmatrix} \right\}$ we have

$$[11, 2] = \frac{\partial a_{12}}{\partial u^1} = 0. \quad \dots (a)$$

Also at the point O , $u^1 = 0$, and we see that, for all small displacements on the surface at O , $ds^2 = (du^1)^2$ no matter what value du^2 is given. Thus for $u^1 = 0$, $a_{12} = a_{22} = 0$. Consequently from (a) it follows that



a_{12} is zero everywhere and the coordinate curves are orthogonal. The line-element is therefore of the form

$$ds^2 = (du^1)^2 + a_{22} (du^2)^2.$$

Moreover if we take small distances OP, OQ , each equal to u^1 , along the two neighbouring geodesics obtained by giving u^2 the values u^2 and $u^2 + du^2$, we see that $PQ = \sqrt{a_{22}} du^2$ and the angle POQ equals du^2 . Also the surface approximates to a plane at O , so that approximately

$$PQ/u^1 = du^2.$$

We conclude that, if $\sqrt{a_{22}}$ is expanded in ascending powers of u^1 , we have

$$\sqrt{a_{22}} = u^1 + \dots,$$

the remaining terms being of higher degree than the first in u^1 .

2. Prove that

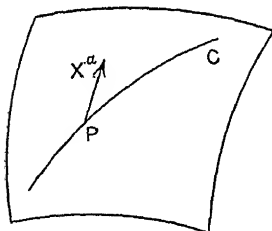
$$\frac{\partial a_{\alpha\beta}}{\partial u^\gamma} = a_{\alpha\delta} \left\{ \begin{matrix} \delta \\ \beta\gamma \end{matrix} \right\} + a_{\beta\delta} \left\{ \begin{matrix} \delta \\ \alpha\gamma \end{matrix} \right\}.$$

3. Deduce from Ex. 2 that if all the Christoffel symbols vanish at O , so do the first derivatives of $a_{\alpha\beta}$, and vice versa.

4. Show that geodesic polar coordinates at O are also geodesic coordinates at O .

7. Parallelism with respect to a surface.

Let C be a curve on the surface and let there be defined a surface vector X^α at each point of C . The coordinates u^α of any point on C are functions of a parameter t and the components of the vector-field X^α are also functions of t .



If we take another coordinate system \bar{u}^α , then the new components \bar{X}^α of the vector-field satisfy the equations of transformation

$$\bar{X}^\alpha = \frac{\partial \bar{u}^\alpha}{\partial u^\sigma} X^\sigma, \dots \quad (43)$$

and are also functions of t . Differentiating with respect to t ,

$$\frac{d\bar{X}^\alpha}{dt} = \frac{\partial \bar{u}^\alpha}{\partial u^\sigma} \frac{dX^\sigma}{dt} + \frac{\partial^2 \bar{u}^\alpha}{\partial u^\sigma \partial u^\tau} X^\sigma \frac{du^\tau}{dt}.$$

If we substitute for $\frac{\partial^2 \bar{u}^\alpha}{\partial u^\sigma \partial u^\tau}$ from (39) we have

$$\frac{d\bar{X}^\alpha}{dt} = \frac{\partial \bar{u}^\alpha}{\partial u^\sigma} \frac{dX^\sigma}{dt} + \left\{ \begin{matrix} \alpha \\ \sigma\tau \end{matrix} \right\} \frac{\partial \bar{u}^\alpha}{\partial u^\sigma} X^\sigma \frac{du^\tau}{dt} - \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} \frac{\partial \bar{u}^\beta}{\partial u^\sigma} \frac{\partial \bar{u}^\gamma}{\partial u^\tau} X^\sigma \frac{du^\tau}{dt}.$$

With the help of (43) we may write this equation

$$\frac{d\bar{X}^\alpha}{dt} + \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} \bar{X}^\beta \frac{d\bar{u}^\gamma}{dt} = \frac{\partial \bar{u}^\alpha}{\partial u^e} \left[\frac{dX^e}{dt} + \left\{ \begin{matrix} e \\ \sigma\tau \end{matrix} \right\} X^\sigma \frac{du^\tau}{dt} \right]. \quad (44)$$

This expresses the fact that

$$\frac{\partial X^\alpha}{\partial t} + \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} X^\beta \frac{du^\gamma}{dt} \quad . \quad . \quad . \quad (45)$$

is a contravariant vector, which we shall call the *surface intrinsic derivative of X^α* .

Let us consider the equation

$$\frac{\partial X^\alpha}{\partial t} + \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} X^\beta \frac{du^\gamma}{dt} = 0. \quad . \quad . \quad . \quad (46)$$

This is a vector equation and therefore if it is satisfied in one coordinate system it is true in every other. Now (46) is a differential equation of the first order and, if we are given the value of X^α at any given point of C , then (46) defines a unique vector X^α at every other point of the curve. Thus we have a unique vector-field defined in this way along C . It will be seen that (46) is exactly similar in form to (6), p. 141, which is the equation satisfied by a system of parallel space vectors along a curve. We shall therefore say that the system of surface vectors defined by (46) is a *system of parallel vectors with respect to the surface*. This is our *definition of parallel vectors with respect to the surface* and it will be noticed that surface vectors are defined as *parallel along a curve on the surface*. Thus if we are given two points A and B of the surface and a vector X^α is given at A then we can define a parallel vector at B only when we are given a curve joining A and B , and in general this parallel vector at B will depend on the particular curve taken.

Expressed in other terms, if we are given a closed circuit on the surface and, starting with a given vector at a point of the circuit, we take the parallel vectors along the curve with respect to the surface, then there is no a priori reason why we should arrive at the same initial vector when we have completed the circuit. In fact, the parallel propagation with respect to a surface of a vector round a closed circuit generally results in a new vector when we have again arrived at the initial point.

EXAMPLES.

1. If X^α is a vector-field satisfying (46), show that $a_{\alpha\beta} X^\alpha X^\beta$ remains constant along the curve. Deduce that *the magnitudes of two parallel vectors are equal*.

$$\left[\text{Use the result that } \frac{da_{\alpha\beta}}{dt} = ([\alpha\gamma, \beta] + [\beta\gamma, \alpha]) \frac{du^\gamma}{dt} \right]$$

2. If X^α, Y^α are two parallel vector-fields show that $a_{\alpha\beta} X^\alpha Y^\beta$ remains constant. Deduce that *the angles between two vectors and their parallels are equal*.

3. If our surface is a plane and rectilinear coordinates are taken in the plane, show that the equation for a parallel vector-field becomes $\frac{dX^\alpha}{dt} = 0$ in such coordinates. In other words, the parallel vectors with respect to the plane as defined by (46) are identical with the ordinary Euclidean parallels in the plane.

4. Prove that the unit tangent vector $\lambda^\alpha = \frac{du^\alpha}{ds}$ of a geodesic G is a parallel vector-field along the geodesic.

$$\left[\text{From (34) the equations of } G \text{ can be written } \frac{\delta \lambda^\alpha}{\delta s} = 0. \right]$$

5. If X^α is a parallel vector-field along a geodesic G , show that it makes a constant angle with G .

[We see from these examples that surface parallelism as we have defined it possesses many of the properties belonging to ordinary Euclidean parallel vectors in space.]

6. If X^α is a parallel vector-field along a curve C , parameter t , show that its covariant components satisfy the equations

$$\frac{\delta X_\alpha}{\delta t} = \frac{dX_\alpha}{dt} - \left\{ \begin{matrix} \gamma \\ \beta \alpha \end{matrix} \right\} X_\gamma \frac{du^\beta}{dt} = 0.$$

8. Intrinsic and covariant differentiation of surface tensors.

We are now in a position to consider the problem which corresponds to that already discussed for space tensors in Chap. XII, p. 140, namely, the problem of forming new tensors by the differentiation of given surface tensor-fields. The method of approach is exactly the same as that of the chapter mentioned. Having defined parallel vector-fields along a curve with respect to our surface the student has merely to read over §§ 2, 3, pp. 143—147, substituting surface tensors for space tensors and surface coordinates for space coordinates. We shall therefore confine ourselves to stating the results for a typical surface tensor.

If $X_{\beta\gamma}^\alpha$ is a surface tensor defined along a curve C , in which case its components are functions of a parameter t , then the quantities

$$\frac{dX_{\beta\gamma}^\alpha}{dt} \equiv \frac{dX_{\beta\gamma}^\alpha}{dt} + \left\{ \begin{matrix} \alpha \\ \sigma\tau \end{matrix} \right\} X_{\beta\gamma}^\sigma \frac{dw^\tau}{dt} - \left\{ \begin{matrix} \sigma \\ \beta\tau \end{matrix} \right\} X_{\sigma\gamma}^\alpha \frac{dw^\tau}{dt} - \left\{ \begin{matrix} \sigma \\ \tau\gamma \end{matrix} \right\} X_{\beta\sigma}^\alpha \frac{dw^\tau}{dt} \quad \dots \dots \dots (47)$$

form a tensor of the same order and type as $X_{\beta\gamma}^\alpha$, and we call it *the intrinsic derivative of $X_{\beta\gamma}^\alpha$ with respect to t* .

Also if $X_{\beta\gamma}^\alpha$ is a tensor-field defined over the surface, its components being hence functions of u^α , the quantities

$$X_{\beta\gamma,\delta}^\alpha \equiv \frac{\partial X_{\beta\gamma}^\alpha}{\partial u^\delta} + \left\{ \begin{matrix} \alpha \\ \sigma\delta \end{matrix} \right\} X_{\beta\gamma}^\sigma - \left\{ \begin{matrix} \sigma \\ \beta\delta \end{matrix} \right\} X_{\sigma\gamma}^\alpha - \left\{ \begin{matrix} \sigma \\ \gamma\delta \end{matrix} \right\} X_{\beta\sigma}^\alpha \quad (48)$$

form a tensor with one covariant index more than $X_{\beta\gamma}^\alpha$. We call it *the covariant derivative of $X_{\beta\gamma}^\alpha$* , and we see that covariant differentiation is indicated by a comma just as in the case of space tensors.

Let us take geodesic coordinates at a given point O of the surface. In such coordinates the Christoffel symbols vanish at the point O , and (47) and (48) immediately show that at O the intrinsic and covariant derivatives of a tensor reduce to the ordinary derivatives. Thus, following the argument of § 4, p. 148, we conclude that the rules of the ordinary differential calculus for finding the derivatives of the sums and products of functions also hold in finding the intrinsic and covariant derivatives of the sums and products of tensors. We leave the student to formulate the rules in the manner of the section quoted*.

Moreover, when we adopt geodesic coordinates at O , the ordinary derivatives of the fundamental tensors $a_{\alpha\beta}$, $a^{\alpha\beta}$ and of the determinant a are all zero. Consequently it follows that *the covariant derivatives of the metric tensors $a_{\alpha\beta}$, $a^{\alpha\beta}$ and of the ε -systems and Kronecker deltas are all zero*, a result similar to that already found for space. These tensors can therefore be regarded as constants when finding the intrinsic or covariant derivatives of any combination of surface tensors.

* The analogy is not quite complete. In the previous case we used rectangular space coordinates in which the Christoffel symbols are *everywhere* zero whereas here the surface Christoffel symbols vanish only at a *given point*.

EXAMPLES.

1. Show that $\frac{d}{dt}(X^\alpha Y_\alpha) = \frac{\partial X^\alpha}{\partial t} Y_\alpha + X^\alpha \frac{\partial Y_\alpha}{\partial t}$.
2. Prove that $X_{\alpha,\beta} - X_{\beta,\alpha} = \frac{\partial X_\alpha}{\partial u^\beta} - \frac{\partial X_\beta}{\partial u^\alpha}$.
3. Show that $\frac{\partial}{\partial x^\alpha}(a_{\beta\gamma} X^\beta Y^\gamma) = X_{\beta,\alpha} Y^\beta + X^\beta Y_{\beta,\alpha}$.
4. Show that the raising or lowering of indices can be performed before or after the operations of intrinsic and covariant derivation.
5. Prove that two indices may be contracted either before or after the operations of intrinsic and covariant derivation.
6. If X is the magnitude of X^α , show that $\frac{\partial X}{\partial u^\alpha} = X_{\beta,\alpha} X^\beta / X$.
7. Writing out in full the tensor equation $\varepsilon_{\alpha\beta,\gamma} = 0$ and putting $\alpha, \beta = 1, 2$, prove that

$$\frac{\partial \log \sqrt{a}}{\partial u^\gamma} = \left\{ \begin{matrix} \beta \\ \beta\gamma \end{matrix} \right\}.$$

9. The Riemann-Christoffel tensor. The Gaussian curvature of a surface.

Let X_α be a covariant vector-field defined over the surface; its covariant derivative is given by

$$X_{\alpha,\beta} = \frac{\partial X_\alpha}{\partial u^\beta} - \left\{ \begin{matrix} \sigma \\ \alpha\beta \end{matrix} \right\} X_\sigma.$$

Since this is also a tensor we can take its covariant derivative and the resulting tensor we call *the second covariant derivative of X_α* . We denote this second covariant derivative by $X_{\alpha,\beta\gamma}$ and we shall see that it is not in general symmetric in β and γ . The procedure is exactly the same as that of p. 152, and we shall leave the student to prove the following results.

We have the formula

$$X_{\alpha,\beta\gamma} - X_{\alpha,\gamma\beta} = R^\delta_{\alpha\beta\gamma} X_\delta, \quad \dots \dots (49)$$

where we have put

$$R^\delta_{\alpha\beta\gamma} = \frac{\partial}{\partial u^\beta} \left\{ \begin{matrix} \sigma \\ \alpha\gamma \end{matrix} \right\} - \frac{\partial}{\partial u^\gamma} \left\{ \begin{matrix} \sigma \\ \alpha\beta \end{matrix} \right\} + \left\{ \begin{matrix} \sigma \\ \alpha\gamma \end{matrix} \right\} \left\{ \begin{matrix} \sigma \\ \beta\sigma \end{matrix} \right\} - \left\{ \begin{matrix} \sigma \\ \alpha\beta \end{matrix} \right\} \left\{ \begin{matrix} \sigma \\ \gamma\sigma \end{matrix} \right\}. \quad (50)$$

Equation (49) shows that $R^\delta_{\alpha\beta\gamma}$ is a tensor and (50) tells us that it depends only on the metric tensor and its derivatives. It is called *the Riemann-Christoffel tensor of the surface*. Also,

if we lower the index δ , we obtain the associated tensor

$$R_{\delta\alpha\beta\gamma} = a_{\delta\alpha} R^{\alpha}_{\cdot\beta\gamma},$$

and it possesses the skew-symmetric properties

$$\begin{aligned} R_{\delta\alpha\beta\gamma} &= -R_{\delta\alpha\gamma\beta}, \\ R_{\delta\alpha\beta\gamma} &= -R_{\alpha\delta\beta\gamma}, \end{aligned}$$

that is, $R_{\delta\alpha\beta\gamma}$ is skew-symmetric in δ, α and in β, γ .

Let us put

$$K = \frac{1}{4} \varepsilon^{\delta\alpha} \varepsilon^{\beta\gamma} R_{\delta\alpha\beta\gamma}, \dots \dots \dots (51)$$

which shows that K is an invariant of the surface. If we multiply this equation by $\varepsilon_{\lambda\mu} \varepsilon_{\sigma\tau}$ we get

$$K \varepsilon_{\lambda\mu} \varepsilon_{\sigma\tau} = \frac{1}{4} \delta^{\delta\alpha}_{\lambda\mu} \delta^{\beta\gamma}_{\sigma\tau} R_{\delta\alpha\beta\gamma}.$$

But, owing to the skew-symmetric properties of $R_{\delta\alpha\beta\gamma}$,

$$\delta^{\delta\alpha}_{\lambda\mu} \delta^{\beta\gamma}_{\sigma\tau} R_{\delta\alpha\beta\gamma} = 4 R_{\lambda\mu\sigma\tau}.$$

Hence

$$R_{\lambda\mu\sigma\tau} = K \varepsilon_{\lambda\mu} \varepsilon_{\sigma\tau}, \dots \dots \dots (52)$$

showing that the Riemann-Christoffel tensor of a surface can be expressed in terms of the invariant K and the ε -systems. This invariant K is called *the total curvature* or *the Gaussian curvature of the surface*. It is of course intrinsic and is a function of the α 's and their derivatives.

EXAMPLES.

1. Show from (52) that $K = R_{1212}/a$. This formula enables us to calculate K in any particular case.

2. Prove that

$$R_{\delta\alpha\beta\gamma} = \frac{\partial}{\partial u^{\beta}} [\alpha\gamma, \delta] - \frac{\partial}{\partial u^{\gamma}} [\alpha\beta, \delta] + \left\{ \begin{matrix} \sigma \\ \alpha\beta \end{matrix} \right\} [\delta\gamma, \sigma] - \left\{ \begin{matrix} \sigma \\ \alpha\gamma \end{matrix} \right\} [\delta\beta, \sigma].$$

3. If the coordinate system is orthogonal, that is, if $a_{12} = 0$, then

$$K = -\frac{1}{2\sqrt{a}} \left[\frac{\partial}{\partial u^1} \left(\frac{1}{\sqrt{a}} \frac{\partial a_{22}}{\partial u^1} \right) + \frac{\partial}{\partial u^2} \left(\frac{1}{\sqrt{a}} \frac{\partial a_{11}}{\partial u^2} \right) \right].$$

[This follows by direct calculation from Exs. 1 and 2.]

4. If the line-element is of the form $ds^2 = (du^1)^2 + a_{22} (du^2)^2$, show that

$$K = -\frac{1}{\sqrt{a_{22}}} \frac{\partial^2 \sqrt{a_{22}}}{(\partial u^1)^2}$$

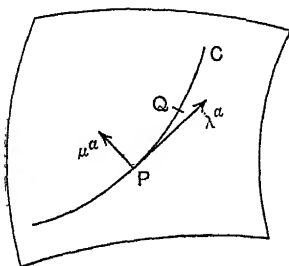
10. The geodesic curvature of a curve on a surface.

Let C be a given curve on our surface and let us take as parameter the arc s of the curve measured from some fixed point on it. The curve is therefore given by equations of the form

$$u^\alpha = u^\alpha(s) \quad \dots \quad (53)$$

and we have the equation

$$a_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = 1 \quad \dots \quad (54)$$



satisfied at every point of the curve.

If P is a point on the curve whose coordinates are u^α and Q is a neighbouring point of C corresponding to an increment ds in the arc, then the infinitesimal vector \overrightarrow{PQ} has for its components du^α . Thus the tangent vector to C , which is the limiting position of the vector \overrightarrow{PQ} as Q tends to P , is given by the vector $\frac{du^\alpha}{ds}$. Moreover this vector is a unit vector as may be seen from (54).

Consequently the unit tangent vector of C is given by

$$\lambda^\alpha \equiv \frac{du^\alpha}{ds} \quad \dots \quad (55)$$

Since λ^α is a unit vector at each point of C , we have

$$a_{\alpha\beta} \lambda^\alpha \lambda^\beta = 1.$$

Differentiating this equation intrinsically with respect to s , we get

$$a_{\alpha\beta} \lambda^\alpha \frac{\delta \lambda^\beta}{\delta s} = 0,$$

which shows that the vector $\frac{\delta \lambda^\alpha}{\delta s}$ is perpendicular to λ^α . Consequently, if we denote by μ^α the unit vector orthogonal to λ^α ,

$$\frac{\delta \lambda^\alpha}{\delta s} = \sigma \mu^\alpha \quad \dots \quad (56)$$

where σ is an invariant. We choose the sense of μ^α such that the rotation $(\lambda^\alpha, \mu^\alpha)$ is positive and reference to p. 170 shows that this is the same as imposing the condition

$$\varepsilon_{\alpha\beta} \lambda^\alpha \mu^\beta = +1. \quad \dots \quad (57)$$

When we have fixed the sense of μ^α in this way, equations (56) determine σ uniquely not only in magnitude but also in sign. The vector μ^α is called the unit normal vector of C and the invariant σ is called the surface or geodesic curvature of the curve.

Owing to the manner in which we have chosen μ^α we have

$$\mu^\alpha = \varepsilon^{\beta\alpha} \lambda_\beta, \quad \lambda^\alpha = \varepsilon^{\alpha\beta} \mu_\beta. \quad (58)$$

Differentiating the first of these intrinsically with respect to s , it becomes

$$\frac{\delta \mu^\alpha}{\delta s} = \varepsilon^{\beta\alpha} \frac{\delta \lambda_\beta}{\delta s} = \varepsilon^{\beta\alpha} \sigma \mu_\beta = -\sigma \varepsilon^{\alpha\beta} \mu_\beta = -\sigma \lambda^\alpha.$$

Consequently we have the two related formulae

$$\left. \begin{aligned} \frac{\delta \lambda^\alpha}{\delta s} &= \sigma \mu^\alpha, \\ \frac{\delta \mu^\alpha}{\delta s} &= -\sigma \lambda^\alpha. \end{aligned} \right\} \quad (59)$$

The student will notice that there is some resemblance between these formulae and the formulae (10), p. 159, connecting the curvature and torsion of a curve in space. We may hence refer briefly to (59) as the *Frenet formulae of the curve C relative to the surface*.

EXAMPLES.

1. If G is the geodesic touching C at P and if equal arc-distances, s , be taken along G and C from P , giving the points Q and R respectively, prove that as s tends to zero the vector \overrightarrow{QR} tends to the normal vector and $(QR/s^2) \rightarrow \sigma/2$.

Let the coordinates of P be u^α . If we expand the coordinates of R in terms of s by Taylor's series we get

$$R \equiv u^\alpha + \left(\frac{du^\alpha}{ds}\right)s + \frac{1}{2}\left(\frac{d^2u^\alpha}{ds^2}\right)s^2 + \dots$$

the remaining terms being of order higher than s^2 . But $\frac{du^\alpha}{ds} = \lambda^\alpha$ and $\frac{d^2u^\alpha}{ds^2} = \frac{d\lambda^\alpha}{ds}$.

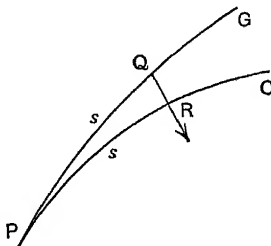
Hence this expansion becomes

$$R \equiv u^\alpha + \lambda^\alpha s + \frac{1}{2} \frac{d\lambda^\alpha}{ds} s^2 + \dots \quad (i)$$

Also expanding the coordinates of Q in powers of s and denoting by bars quantities referring to the geodesic G , we have

$$Q \equiv u^\alpha + \left(\frac{\bar{d}u^\alpha}{ds}\right)s + \frac{1}{2}\left(\frac{\bar{d}^2u^\alpha}{ds^2}\right)s^2 + \dots$$

Now, since the curves touch at P , $\left(\frac{\bar{d}u^\alpha}{ds}\right) = \lambda^\alpha$ and from the equation



of the geodesic we have

$$\frac{d^2 u^\alpha}{ds^2} = - \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} \left(\frac{du^\beta}{ds} \right) \left(\frac{du^\gamma}{ds} \right) = - \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} \lambda^\beta \frac{du^\gamma}{ds}$$

at the point P . Hence

$$Q \equiv u^\alpha + \lambda^\alpha s - \frac{1}{2} \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} \lambda^\beta \frac{du^\gamma}{ds} s^2 + \dots \quad (ii)$$

Subtracting (ii) from (i) we get for the vector \overrightarrow{QR} when s is small

$$\overrightarrow{QR} = \frac{1}{2} \left(\frac{d\lambda^\alpha}{ds} + \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} \lambda^\beta \frac{du^\gamma}{ds} \right) s^2 + \dots,$$

that is,

$$\lim_{s \rightarrow 0} \frac{\overrightarrow{QR}}{s^2} = \frac{1}{2} \frac{\delta \lambda^\alpha}{\delta s} = \frac{1}{2} \sigma \mu^\alpha$$

and our result follows immediately.

2. Prove that $\sigma^2 = a_{\alpha\beta} \frac{\delta \lambda^\alpha}{\delta s} \frac{\delta \lambda^\beta}{\delta s}$.

3. Prove that $\sigma = \varepsilon_{\alpha\beta} \lambda^\alpha \frac{\delta \lambda^\beta}{\delta s} = - \varepsilon_{\alpha\beta} \frac{\delta \lambda^\alpha}{\delta s} \lambda^\beta$.

4. If C is a geodesic show that $\sigma = 0$, and conversely if $\sigma = 0$ then the curve is a geodesic. *Thus geodesics are curves whose geodesic curvatures are zero.*

[The equation of a geodesic can be written $\frac{\delta \lambda^\alpha}{\delta s} = 0$.]

5. Prove that $\frac{d^2 u^\alpha}{ds^2} + \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} = - \sigma \varepsilon^{\alpha\beta} a_{\beta\gamma} \frac{du^\gamma}{ds}$.

6. Use Ex. 5 to deduce that the geodesic curvature σ_1 of the u^1 -curves is given by

$$\sigma_1 = \frac{1}{\sqrt{a}} \frac{\partial}{\partial u^1} \left(\frac{a_{12}}{\sqrt{a_{11}}} \right) - \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a_{11}}}{\partial u^2}$$

and find the similar formula for the geodesic curvature σ_2 of the u^2 -curves.

[For the u^1 -curves we have $\frac{du^\alpha}{ds} = \left(\frac{1}{\sqrt{a_{11}}}, 0 \right)$.]

7. When the coordinate curves are orthogonal prove that

$$\sigma_1 = - \frac{1}{2 \sqrt{a_{22}}} \frac{\partial (\log a_{11})}{\partial u^2}, \quad \sigma_2 = \frac{1}{2 \sqrt{a_{11}}} \frac{\partial (\log a_{22})}{\partial u^1}.$$

11. Beltrami's differential parameters.

If φ and ψ are two invariant functions of position on the surface and if we denote by $\varphi_{,\alpha}$ and $\psi_{,\alpha}$ their derivatives with respect to u^α , that is,

$$\varphi_{,\alpha} = \frac{\partial \varphi}{\partial u^\alpha}, \quad \psi_{,\alpha} = \frac{\partial \psi}{\partial u^\alpha}, \quad \dots \quad (60)$$

then these derivatives are covariant vectors. Consequently the quantity

$$\nabla(\varphi, \psi) \equiv \alpha^{\alpha\beta} \varphi_{,\alpha} \psi_{,\beta} \quad (61)$$

is an invariant, and is often referred to as *Beltrami's mixed differential parameter of φ, ψ* .

If ψ is put equal to φ then (61) gives the invariant

$$\nabla \varphi = \nabla(\varphi, \varphi) = \alpha^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta}, \quad \dots \quad (62)$$

which is called *Beltrami's first differential parameter of φ* , and the symbol $\nabla \varphi$ is read "Nabla φ ".

Again, if X^α is a contravariant surface vector, we may take its covariant derivative and contract the two indices, giving $X^\alpha_{,\alpha}$. This is obviously an invariant and, writing it out in full, it reads

$$X^\alpha_{,\alpha} = \frac{\partial X^\alpha}{\partial u^\alpha} + \left\{ \begin{matrix} \alpha \\ \beta \alpha \end{matrix} \right\} X^\beta \dots \dots \dots (63)$$

This invariant may be called *the surface divergence of X^α* in analogy to (23) of p. 151. Using the result (p. 182) that

$$\left\{ \begin{matrix} \alpha \\ \beta \alpha \end{matrix} \right\} = \frac{1}{\sqrt{a}} \frac{\partial \sqrt{a}}{\partial u^\beta}$$

the divergence of X^α may be written

$$X^\alpha_{,\alpha} = \frac{1}{\sqrt{a}} \frac{\partial}{\partial u^\alpha} (\sqrt{a} X^\alpha), \quad \dots \dots \dots (64)$$

which is the formula of most use for calculation.

In the particular case where X^α is the vector $\alpha^{\alpha\beta} \varphi_{,\beta}$, φ being an invariant function of position, its divergence reads

$$X^\alpha_{,\alpha} = \alpha^{\alpha\beta} \varphi_{,\beta\alpha} = \alpha^{\alpha\beta} \varphi_{,\alpha\beta}.$$

This invariant is called *Beltrami's second differential parameter of φ* and is often denoted by $\Delta \varphi$. Thus*

$$\Delta \varphi = \alpha^{\alpha\beta} \varphi_{,\alpha\beta} \dots \dots \dots (65)$$

Using (64) it may be written in the equivalent form

$$\Delta \varphi = \frac{1}{\sqrt{a}} \frac{\partial}{\partial u^\alpha} \left(\sqrt{a} \alpha^{\alpha\beta} \frac{\partial \varphi}{\partial u^\beta} \right) \dots \dots \dots (66)$$

* The notation used here for the first and second differential parameters is that adopted by Blaschke, *Vorlesungen über Differentialgeometrie*. These differential parameters are sometimes also denoted by $\Delta_1 \varphi$ and $\Delta_2 \varphi$.

EXAMPLES.

1. Prove that

$$\nabla(u^1) = \frac{a_{22}}{a}, \quad \nabla(u^2) = \frac{a_{11}}{a}, \quad \nabla(u^1, u^2) = -\frac{a_{12}}{a}$$

2. Prove that

$$\Delta(u^1) = \frac{1}{\sqrt{a}} \left[\frac{\partial}{\partial u^1} \left(\frac{a_{22}}{\sqrt{a}} \right) - \frac{\partial}{\partial u^2} \left(\frac{a_{12}}{\sqrt{a}} \right) \right],$$

$$\Delta(u^2) = \frac{1}{\sqrt{a}} \left[\frac{\partial}{\partial u^2} \left(\frac{a_{11}}{\sqrt{a}} \right) - \frac{\partial}{\partial u^1} \left(\frac{a_{12}}{\sqrt{a}} \right) \right].$$

3. Prove that the angle θ between the curves $\varphi = \text{const.}$, $\psi = \text{const.}$, is given by

$$\cos \theta = \frac{-\nabla(\varphi, \psi)}{\sqrt{\nabla\varphi \cdot \nabla\psi}}$$

Deduce that the two curves are orthogonal if $\nabla(\varphi, \psi) = 0$.[This result may be proved by taking as coordinate curves $u^1 \equiv \varphi$, $u^2 \equiv \psi$ and using Ex. 1.]4. *Beltrami's formula for the geodesic curvature of a curve.* If a curve is defined by the relation $\varphi = \text{const.}$, prove that its geodesic curvature is given by

$$\sigma = -\frac{\Delta\varphi}{\sqrt{\nabla\varphi}} - \nabla\left(\varphi, \frac{1}{\sqrt{\nabla\varphi}}\right)$$

[This is an invariant equation. Let us take as special coordinate system $u^2 \equiv \varphi$ and the coordinate curves orthogonal. The given curve is then a u^1 -curve. Use Ex. 1 and 2 of this section and Ex. 7, p. 186.]

12. Green's theorem on a surface.

Let us now consider a closed circuit C on the surface. We shall first define the positive sense of description of the circuit. If we take the simple circuit defined by the curves $u^1 = 0$, $u^2 = 0$, $u^1 = 1$, $u^2 = 1$, we define as the *positive sense* of description that in which the curves are described in the order named, and the opposite sense is negative. The sense of description of every other circuit is positive or negative according as it is or is not the same as the positive description of the above simple circuit.

Denoting the interior of C by S , we know from Green's theorem that, if P and Q are functions of position,

$$\left. \begin{aligned} \frac{\partial P}{\partial u^1} du^1 du^2 &= P \frac{du^2}{ds} ds, \\ \int_S \frac{\partial Q}{\partial u^2} du^1 du^2 &= \int_C Q \frac{du^1}{ds} ds, \end{aligned} \right\} \dots \dots \dots (67)$$

the sense of description of C in the contour integrals being positive.

Now let X^α be a vector-field defined throughout S and let dS be the element of area, that is,

$$dS = \sqrt{a} \, du^1 du^2.$$

Then

$$\iint_S X^\alpha_{,\alpha} dS = \iint_S X^\alpha_{,\alpha} \sqrt{a} \, du^1 du^2.$$

But from (64)

$$X^\alpha_{,\alpha} = \frac{1}{\sqrt{a}} \frac{\partial}{\partial u^\alpha} (\sqrt{a} X^\alpha).$$

Hence

$$\begin{aligned} \iint_S X^\alpha_{,\alpha} dS &= \iint_S \frac{\partial}{\partial u^\alpha} (\sqrt{a} X^\alpha) du^1 du^2 \\ &= \int_C \left[\sqrt{a} (X^1 \frac{du^2}{ds} - X^2 \frac{du^1}{ds}) \right] ds = \int_C \varepsilon_{\alpha\beta} X^\alpha \frac{du^\beta}{ds} ds. \end{aligned}$$

Now if we denote by μ^α the unit inward normal vector of C its covariant components are given (p. 185) by

$$\mu_\alpha = -\varepsilon_{\alpha\beta} \frac{du^\beta}{ds}.$$

Therefore Green's formula takes the form

$$\iint_S X^\alpha_{,\alpha} dS = - \int_C X_\alpha \mu^\alpha ds. \quad \dots \dots (68)$$

In this result let us put

$$X^\alpha = a^{\alpha\beta} \varphi_{,\beta} \psi,$$

where φ, ψ are invariant functions of position. Then

$$X^\alpha_{,\alpha} = a^{\alpha\beta} \varphi_{,\beta} \psi_{,\alpha} + a^{\alpha\beta} \varphi_{,\beta} \psi_{,\alpha} = \nabla(\varphi, \psi) + \psi \Delta \varphi,$$

and (68) becomes

$$\iint_S \nabla(\varphi, \psi) dS = - \int_C \psi(\varphi_{,\alpha} \mu^\alpha) ds - \iint_S \psi \Delta \varphi dS. \quad \dots (69)$$

EXAMPLES.

1. Prove that $\iint_S (\varphi \Delta \psi - \psi \Delta \varphi) dS + \int_C (\varphi \psi_{,\alpha} - \psi \varphi_{,\alpha}) \mu^\alpha ds = 0$.
2. Prove that $\iint_S \nabla \varphi dS = - \int_C \varphi(\varphi_{,\alpha} \mu^\alpha) ds - \iint_S \varphi \Delta \varphi dS$.
3. Deduce from (67) that, if X_α is a covariant vector-field, then

$$\int_C X_\alpha \frac{du^\alpha}{ds} ds = - \iint_S \varepsilon^{\alpha\beta} X_{\alpha,\beta} dS.$$

EXAMPLES XIV.

1. If ω is the angle between the coordinate curves prove that

$$\tan \omega = \frac{\sqrt{a}}{a_{12}}$$

2. If a double system of curves on the surface is defined by the equation $b_{\alpha\beta} du^\alpha dw^\beta = 0$, show that the angle between them is given by

$$\tan \theta = \frac{2 \sqrt{a} \sqrt{b}}{\sqrt{a} a^{\alpha\beta} b_{\alpha\beta}}.$$

Deduce that the double system is orthogonal if $a^{\alpha\beta} b_{\alpha\beta} = 0$.

[Note that the quantity equated to $\tan \theta$ is an absolute invariant, take the double system as coordinate curves, i. e. $b_{11} = b_{22} = 0$, and use Ex. 1.]

3. Let us take a curve C on the surface and take the family of geodesics cutting it orthogonally. If we take this family of geodesics as the u^1 -curves, u^1 measuring the distance along each geodesic from the curve C , show that the line-element is of the form

$$ds^2 = (du^1)^2 + a_{22} (du^2)^2.$$

4. *Gaussian geodesic coordinates.* If in Ex. 3 we take C to be a geodesic and take as u^2 the distance measured along C , then $a_{22}(u^1, u^2)$ of Ex. 3 satisfies the relations, for all values of u^2 ,

$$a_{22}(0, u^2) = 1, \quad \left[\frac{\partial a_{22}}{\partial u^1} \right]_{u^1=0} = 0.$$

5. If the Gaussian curvature of a surface is zero and we use Gaussian geodesic coordinates, show that the line-element is

$$ds^2 = (du^1)^2 + (du^2)^2,$$

which is the same as that of a plane in rectangular Cartesian coordinates.

[Use Ex. 4, p. 183, together with the conditions of the last example.]

6. If K is not zero but a constant, show that in Gaussian geodesic coordinates

$$ds^2 = (du^1)^2 + \cos^2(\sqrt{K}u^1) (du^2)^2, \quad (K > 0),$$

$$ds^2 = (du^1)^2 + \cosh^2(\sqrt{-K}u^1) (du^2)^2, \quad (K < 0).$$

7. If μ^α is the unit normal to a family of curves prove that the geodesic curvature σ of the family is given by

$$\sigma = -\mu^\alpha_{;\alpha}.$$

[May be proved by taking the family as u^1 -curves and the coordinate system orthogonal, in which case $\mu^\alpha = \left(0, \frac{1}{\sqrt{a_{22}}}\right)$.]

8. If the coordinate curves are orthogonal and the tangent vector of a curve C makes an angle θ with the curve $u^2 = \text{const.}$, prove that the geodesic curvature of C is given by

$$\sigma = \frac{d\theta}{ds} + \sigma_1 \cos \theta + \sigma_2 \sin \theta.$$

9. If the line-element is $ds^2 = (du^1)^2 + a_{22}(du^2)^2$ and C is a geodesic, show that the formula of Ex. 8 reads

$$\frac{d\theta}{ds} = -\sigma, \sin \theta.$$

Deduce that

$$\frac{d\theta}{ds} = -\frac{\partial \sqrt{a_{22}}}{\partial u^1} \frac{\sin \theta}{\sqrt{a_{22}}}$$

10. *Laquerre's formula.* If σ_1, σ_2 are the geodesic curvatures of the coordinate curves and ω is the angle between them, show that

$$K = \frac{1}{\sqrt{a}} \left\{ \frac{\partial^2 \omega}{\partial u^1 \partial u^2} - \frac{\partial}{\partial u^1} (\sigma_2 \sqrt{a_{22}}) + \frac{\partial}{\partial u^2} (\sigma_1 \sqrt{a_{11}}) \right\}$$

11. If we have two curves intersecting orthogonally at a point P and ds_1, ds_2 are the elements of length along these curves, and σ_1, σ_2 their geodesic curvatures, prove that

$$\Delta \varphi = \frac{d^2 \varphi}{ds_1^2} + \frac{d^2 \varphi}{ds_2^2} - \sigma_1 \frac{d\varphi}{ds_2} + \sigma_2 \frac{d\varphi}{ds_1}.$$

12. *Parallel propagation round a closed circuit.* If, starting with a given vector at the initial point, we propagate the vector as a parallel vector field round a closed circuit, prove that the final vector on completion of the circuit makes with the original vector an angle equal to $\int \int K dS$ taken throughout the interior of the circuit.

[Let us take the coordinate system such that the line-element is given by $ds^2 = (du^1)^2 + a_{22}(du^2)^2$, and let A^α be the given parallel vector-field, which we may take to be a unit vector-field. If A^α at any point makes an angle θ with the u^1 -curve through the point, then

$$A^1 = \cos \theta, \quad A^2 = \frac{\sin \theta}{\sqrt{a_{22}}}.$$

Since A^α is a parallel vector-field $\frac{\delta A^\alpha}{\delta s} = 0$. Now

$$\begin{aligned} \frac{\delta A^1}{\delta s} &= \frac{d}{ds} (\cos \theta) + \left\{ \begin{matrix} 1 \\ \mu \nu \end{matrix} \right\} A^\mu \frac{du^\nu}{ds} = -\sin \theta \frac{d\theta}{ds} + \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} A^2 \frac{du^2}{ds} \\ &= -\sin \theta \frac{d\theta}{ds} - \frac{1}{2} \frac{\partial a_{22}}{\partial u^1} \frac{\sin \theta}{\sqrt{a_{22}}} \frac{du^2}{ds} \end{aligned}$$

Equating this to zero, we get

$$\frac{d\theta}{ds} = -\frac{\partial \sqrt{a_{22}}}{\partial u^1} \frac{du^2}{ds},$$

and when we integrate this round the contour we obtain for the required angle α the value

$$\begin{aligned}\alpha &= \int_C \frac{d\theta}{ds} ds = - \int_C \frac{\partial \sqrt{a_{22}}}{\partial u^1} \frac{du^2}{ds} ds = - \iint_S \frac{\partial^2 \sqrt{a_{22}}}{(\partial u^1)^2} du^1 du^2 \\ &= - \iint_S \frac{1}{\sqrt{a_{22}}} \frac{\partial^2 \sqrt{a_{22}}}{(\partial u^1)^2} dS.\end{aligned}$$

Thus (p. 183)

$$\alpha = \iint_S K dS.]$$

13. *Gauss' theorem.* Show geometrically that, when a vector is given a parallel displacement round the sides of a geodesic triangle ABC , the angle α of Ex. 12 equals $A + B + C - \pi$. Hence

$$E = A + B + C - \pi = \iint_S K dS.$$

E is called the excess of the triangle.

14. *Bonnet's theorem.* If σ is the geodesic curvature of a curve C forming a continuous closed circuit, prove that

$$\int_C \sigma ds + \iint_S K dS = 2\pi.$$

[Instead of taking A^α of Ex. 12 as a parallel vector-field, take it as the unit tangent vector of C . Observe that $\frac{\delta A^1}{\delta s}$ now equals $-\sigma \sin \theta$ by the surface Frenet formulae and in this case $\alpha = 2\pi$.]

15. If the circuit C has corners at which the exterior angles are $\omega_1, \omega_2, \omega_3$, etc., show that the formula of Ex. 14 is modified in accordance with the result

$$\sum \omega + \int_C \sigma ds + \iint_S K dS = 2\pi.$$

16. Deduce from Ex. 15 that for a surface triangle ABC

$$\int_C \sigma ds + \iint_S K dS = A + B + C - \pi.$$

[This is a generalisation of Ex. 13.]

17. If $\Delta\varphi$ is the angle between two neighbouring geodesic tangents to a curve C at points distant Δs apart, then the geodesic curvature of C is given by

$$\sigma = \text{Lt}_{\Delta s \rightarrow 0} \frac{\Delta\varphi}{\Delta s}.$$

[This is an immediate consequence of Ex. 16 applied to the elementary triangle formed by the two geodesic tangents and the curve.]

18. If we have an infinitesimal circuit ΔS and $\Delta\alpha$ is the angle between the initial and final positions of a vector which is given a parallel displacement round it, prove that the Gaussian curvature of the surface is given by

$$K = \text{Lt}_{\Delta S \rightarrow 0} \frac{\Delta\alpha}{\Delta S}.$$

CHAPTER XV

THE FUNDAMENTAL FORMULAE OF A SURFACE

1. Notation.

In the last chapter we have developed the *intrinsic* properties of a surface, that is, those properties which can be described without reference to the surrounding space. We are now going to investigate the properties of a surface in its relation to the surrounding space.

Consequently in this chapter we are dealing with two distinct systems of coordinates, namely, the three curvilinear coordinates for the surrounding space which we denote by

$$x^r \ (r = 1, 2, 3),$$

and the two curvilinear coordinates of the surface which we denote by

$$u^\alpha \ (\alpha = 1, 2).$$

We shall understand that the following conventions hold as regards the indices:

(i) *Italic indices*, if repeated, are to be summed from 1 to 3 and, if free, are to have the range of values 1 to 3.

(ii) *Greek indices*, if repeated, are to be summed from 1 to 2 and, if free, are to have the range of values 1 to 2.

There will now be tensors of the surrounding space and also tensors of the surface. The former will be indicated by italic indices and the latter by Greek indices. Moreover in this chapter and the next we shall be dealing with quantities which will be tensors both of the surrounding space and of the surface; such quantities will possess both Greek and Roman indices. Thus a set of quantities represented by A_a^{rs} would form a set of *double contravariant tensors in the x -coordinates* and also a set of *covariant vectors in the u -coordinates*. Again B^r would represent a contravariant vector in the x 's and a set of invariants in the u 's, and so on. The notation will become clearer as we proceed.

It is obvious that to examine tensorial character with respect to the x -coordinates we have merely to keep the u -coordinates fixed and change the space coordinates; and to examine tensorial character with respect to the u -coordinates we have merely to fix the x 's and transform the u 's to new surface coordinates.

EXAMPLES.

1. If we fix the u -coordinates and transform x^r to \bar{x}^r show that the new components of the tensor $A_{\alpha\beta}^{rs}$ satisfy the equations

$$\bar{A}_{\alpha\beta}^{rs} = A_{\alpha\beta}^{mn} \frac{\partial \bar{x}^r}{\partial x^m} \frac{\partial \bar{x}^s}{\partial x^n}.$$

2. If we fix the x -coordinates and transform u^α to \bar{u}^α show that the new components are

$$\bar{A}_{\alpha\beta}^{rs} = A_{\sigma\tau}^{rs} \frac{\partial u^\sigma}{\partial \bar{u}^\alpha} \frac{\partial u^\tau}{\partial \bar{u}^\beta}.$$

3. If we transform both systems x^r, u^α to new systems $\bar{x}^r, \bar{u}^\alpha$ simultaneously the new components of $A_{\alpha\beta}^{rs}$ are given by

$$\bar{A}_{\alpha\beta}^{rs} = A_{\sigma\tau}^{mn} \frac{\partial \bar{x}^r}{\partial x^m} \frac{\partial \bar{x}^s}{\partial x^n} \frac{\partial u^\sigma}{\partial \bar{u}^\alpha} \frac{\partial u^\tau}{\partial \bar{u}^\beta}.$$

2. The tangent vectors to a surface.

The equations of a surface may be written

$$x^r = x^r(u^1, u^2). \quad (1)$$

If we take a small displacement du^α on the surface the corresponding components of the displacement in space are given by

$$dx^r = \frac{\partial x^r}{\partial u^\alpha} du^\alpha. \quad (2)$$

Now dx^r is a space vector and it is also a surface invariant, that is, its components are unaltered if the u -coordinates alone are transformed. Similarly du^α is a surface vector and is also a space invariant. Hence if we regard (2) first from the point of view of a transformation of space coordinates and then from the point of view of a transformation of surface coordinates, we see that $\frac{\partial x^r}{\partial u^\alpha}$ is a contravariant space vector and also a covariant surface vector, so that it may be appropriately represented by

$$x_a^r \equiv \frac{\partial x^r}{\partial u^\alpha}. \quad (3)$$

Corresponding to every direction $\frac{du^\alpha}{ds}$ on the surface, the

same direction in the space-coordinates is seen from (2) to be

$$\frac{dx^r}{ds} = x^r_\alpha \frac{du^\alpha}{ds} \quad (4)$$

Moreover every vector defines a direction and a magnitude. Consequently for each surface-vector A^α we have the space-vector

$$A^r = x^r_\alpha A^\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and these two vectors define the same direction and the same magnitude. The vector A^r whose direction is tangent to the surface may be called a *tangent vector of the surface*.

3. The first groundform of a surface.

The line-element in space is given by

$$ds^2 = g_{mn} dx^m dx^n \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and that on the surface is

$$ds^2 = a_{\alpha\beta} du^\alpha du^\beta \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we take a displacement on the surface and therefore also in space the two values for ds^2 must be equal. But the connection between dx^r and du^α in the same displacement is given by (2). Therefore we must have

$$g_{mn} dx^m dx^n = g_{mn} x^m_\alpha x^n_\beta du^\alpha du^\beta = a_{\alpha\beta} du^\alpha du^\beta$$

for all surface displacements du^α . That is,

$$a_{\alpha\beta} = g_{mn} x^m_\alpha x^n_\beta \quad . \quad . \quad . \quad . \quad . \quad (8)$$

since both $a_{\alpha\beta}$ and g_{mn} are symmetric in their indices. Thus if we know g_{mn} and the equations of our surface we can immediately write down the $a_{\alpha\beta}$.

The differential quadratic form

$$A \equiv a_{\alpha\beta} du^\alpha du^\beta \quad . \quad . \quad . \quad . \quad . \quad (9)$$

is called the *first groundform of the surface* and we denote it briefly by A . It obviously gives the square of the length of an elementary displacement du^α on the surface.

EXAMPLES.

1. Show that $\frac{1}{\sqrt{a_{11}}} x_1^r$ and $\frac{1}{\sqrt{a_{22}}} x_2^r$ are the unit space-vectors tangent to the u^1 -curves and u^2 -curves respectively.

2. Show that $a_{12}/\sqrt{a_{11} a_{22}}$ is the cosine of the angle between the coordinate curves on the surface.

3. Show that the magnitudes of A^r and A^α where

$$A^r = x_\alpha^r A^\alpha$$

are equal and deduce that these vectors define the same direction on the surface.

4. Prove $A_\alpha = x_\alpha^r A_r$.

4. The normal vector to the surface.

We shall now find an expression for the normal vector to a surface at any point on it.

Let X^α, Y^α be two surface vectors which are such that the rotation (X^α, Y^α) is *positive*, that is, this rotation is the same as that of the u^1 -curves to the u^2 -curves. The corresponding space components of these vectors are

$$X^r = x_\alpha^r X^\alpha, \quad Y^r = x_\alpha^r Y^\alpha. \quad \dots \dots (10)$$

The unit normal vector to the surface is orthogonal to both X^r and Y^r , which are tangent vectors. Hence if we denote the *unit normal vector* by ξ^r its covariant components are given by

$$\xi_r (X Y \sin \theta) = \varepsilon_{rst} X^s Y^t \quad \dots \dots (11)$$

where X, Y are the magnitudes of X^r, Y^r and θ is the angle between them. We note that the direction of ξ^r is such that the *space orientation* of (X^r, Y^r, ξ^r) is *positive*.

We deduce from (11) that

$$\xi_r (X Y \sin \theta) = \varepsilon_{rst} x_\alpha^s x_\beta^t X^\alpha Y^\beta.$$

But

$$X Y \sin \theta = \varepsilon_{\alpha\beta} X^\alpha Y^\beta$$

since the rotation of (X^α, Y^α) is positive. Hence

$$(\xi_r \varepsilon_{\alpha\beta} - \varepsilon_{rst} x_\alpha^s x_\beta^t) X^\alpha Y^\beta = 0$$

and this is true for all values of the surface vectors X^α, Y^α .

Consequently

$$\xi_r \varepsilon_{\alpha\beta} = \varepsilon_{rst} x_\alpha^s x_\beta^t. \quad (12)$$

Multiplying this equation across by $\varepsilon^{\alpha\beta}$ we conclude that

$$\xi_r = \frac{1}{2} \varepsilon^{\alpha\beta} \varepsilon_{rst} x_\alpha^s x_\beta^t, \quad (13)$$

which shows clearly that ξ_r is a covariant space vector and a surface invariant.

EXAMPLES.

1. Show that $\xi_r x_\alpha^r = 0$.

2. Show that the vectors (x_1^r, x_2^r, ξ^r) have the same orientation as the tangents to the x -coordinate curves, that is, the orientation is positive.

3. Prove that

$$\varepsilon^{\alpha\beta} x_\alpha^r x_\beta^s = \varepsilon^{mrs} \xi_m.$$

4. *Special coordinate systems.* If we choose the space coordinates to be orthogonal and such that $x^3 = 0$ is the given surface and choose the u -curves on the surface to be the intersections of the surface with the x^1 - and x^2 -surfaces, show that we have the following relations for all points of the surface:

$$(i) \quad x_1^r = \delta_1^r, \quad x_2^r = \delta_2^r;$$

$$(ii) \quad a_{\alpha\beta} = g_{\alpha\beta}, \quad g_{\alpha 3} = 0; \quad a^{\alpha\beta} = g^{\alpha\beta}, \quad g^{\alpha 3} = 0; \quad g^{33} = \frac{1}{g_{33}};$$

$$(iii) \quad \xi^r = \left(0, 0, \frac{1}{\sqrt{g_{33}}} \right).$$

5. By taking the special coordinate systems of Ex. 4 prove the tensor equation

$$a^{\alpha\beta} x_\alpha^r x_\beta^s = g^{rs} - \xi^r \xi^s.$$

6. *Projection of a vector on the surface.* From Ex. 5 deduce that the projection of a vector X^r on the surface is the vector $a^{\alpha\beta} x_\alpha^r x_\beta^m X_m$.

[The projection of X^r on the surface is $\{X^r - \xi^r (\xi^s X_s)\}$.]

5. The tensor derivation of tensors.

Our next problem is the formation of new tensors by the differentiation of given tensor-fields, and we shall approach it by the same methods as we have already used (see pp. 146, 180).

We first consider a curve C lying on the given surface and along it we take a parameter t . If X^r is a space vector

which is defined along C and which forms a parallel field *with respect to space*, then we know (p. 141) that the vector satisfies the relation

$$\frac{\delta X^r}{\delta t} = \frac{dX^r}{dt} + \begin{matrix} r \\ mn \end{matrix} X^m \frac{dx^n}{dt} = 0. \quad . \quad . \quad (14)$$

Here the Christoffel symbol refers to the x 's, i.e. to the space coordinates and is written in italic letters. To avoid confusion with the Christoffel symbols of the surface we place a g after the symbol to show that it is formed from the g_{mn} . The corresponding equation satisfied by the covariant components X_r of a parallel field is

$$\frac{\delta X_r}{\delta t} = \frac{dX_r}{dt} - \begin{matrix} m \\ rn \end{matrix} X_m \frac{dx^n}{dt} = 0. \quad . \quad . \quad (15)$$

Again, if X^α is a surface vector which is defined along C and which forms a parallel field *with respect to the surface*, we know (p. 179) that X^α satisfies

$$\frac{\delta X^\alpha}{\delta t} = \frac{dX^\alpha}{dt} + \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\}_a X^\beta \frac{du^\gamma}{dt} = 0, \quad . \quad . \quad . \quad (16)$$

and the covariant components X_α satisfy

$$\frac{\delta X_\alpha}{\delta t} = \frac{dX_\alpha}{dt} - \left\{ \begin{matrix} \beta \\ \alpha \gamma \end{matrix} \right\}_a X_\beta \frac{du^\gamma}{dt} = 0. \quad . \quad . \quad . \quad (17)$$

The Christoffel symbols in these formulae refer to the surface and are written in Greek letters. We also use a subscript a to show that they are formed from the $a_{\alpha\beta}$.

Let us now take a tensor-field defined along C , for example, X^α_β , which is a contravariant space vector and a double covariant surface tensor. We take A_r a covariant space vector-field along C parallel with respect to space and B^α, C^α two contravariant surface vector-fields parallel with respect to the surface. Then $X^\alpha_\beta A_r B^\alpha C^\beta$ is an invariant function of t . Its derivative with respect to t is therefore also an invariant with respect to both the space and surface coordinates. But when we use the equations of parallelism we get

$$\begin{aligned}
 \frac{d}{dt}(X_{\alpha\beta}^r A_r B^\alpha C^\beta) &= \frac{dX_{\alpha\beta}^r}{dt} A_r B^\alpha C^\beta + X_{\alpha\beta}^r \frac{dA_r}{dt} B^\alpha C^\beta \\
 &\quad + X_{\alpha\beta}^r A_r \frac{dB^\alpha}{dt} C^\beta + X_{\alpha\beta}^r A_r B^\alpha \frac{dC^\beta}{dt} \\
 &= \left\{ \frac{dX_{\alpha\beta}^r}{dt} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\}_g X_{\alpha\beta}^m \frac{dx^n}{dt} - \left\{ \begin{matrix} \sigma \\ \alpha\tau \end{matrix} \right\}_a X_{\sigma\beta}^r \frac{du^\tau}{dt} \right. \\
 &\quad \left. - \left\{ \begin{matrix} \sigma \\ \beta\tau \end{matrix} \right\}_a X_{\alpha\sigma}^r \frac{du^\tau}{dt} \right\} A_r B^\alpha C^\beta.
 \end{aligned}$$

This is an invariant for all values of the vectors A_r, B^α, C^α . Consequently by the quotient law we conclude that

$$\begin{aligned}
 \frac{dX_{\alpha\beta}^r}{dt} &\equiv \frac{dX_{\alpha\beta}^r}{dt} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\}_g X_{\alpha\beta}^m \frac{dx^n}{dt} - \left\{ \begin{matrix} \sigma \\ \alpha\tau \end{matrix} \right\}_a X_{\sigma\beta}^r \frac{du^\tau}{dt} \\
 &\quad - \left\{ \begin{matrix} \sigma \\ \beta\tau \end{matrix} \right\}_a X_{\alpha\sigma}^r \frac{du^\tau}{dt} \dots \dots \dots (18)
 \end{aligned}$$

is a tensor of the same type as $X_{\alpha\beta}^r$. We shall call it the *intrinsic derivative of $X_{\alpha\beta}^r$ with respect to t* .

If $X_{\alpha\beta}^r$ is defined over the whole surface, its components are functions of the u 's and we have a tensor-field over the surface. Also if C is any curve on the surface

$$\frac{\partial X_{\alpha\beta}^r}{\partial t} = \left[\frac{\partial X_{\alpha\beta}^r}{\partial u^\gamma} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\}_g X_{\alpha\beta}^m x_\gamma^n - \left\{ \begin{matrix} \sigma \\ \alpha\gamma \end{matrix} \right\}_a X_{\sigma\beta}^r - \left\{ \begin{matrix} \sigma \\ \beta\gamma \end{matrix} \right\}_a X_{\alpha\sigma}^r \right] \frac{du^\gamma}{dt}.$$

Since $\frac{du^\gamma}{dt}$ is an arbitrary surface vector owing to the arbitrariness of C and since the right-hand side of this last equation is a tensor, we conclude that

$$X_{\alpha\beta,\gamma}^r = \frac{\partial X_{\alpha\beta}^r}{\partial u^\gamma} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\}_g X_{\alpha\beta}^m x_\gamma^n - \left\{ \begin{matrix} \sigma \\ \alpha\gamma \end{matrix} \right\}_a X_{\sigma\beta}^r - \left\{ \begin{matrix} \sigma \\ \beta\gamma \end{matrix} \right\}_a X_{\alpha\sigma}^r \quad (19)$$

is a tensor which is singly contravariant in the space-coordinates and triply covariant in the surface-coordinates. We shall call it the *tensor derivative of $X_{\alpha\beta}^r$ with respect to the u 's*.

Any other type of tensor may be treated in the same way and the corresponding forms for the intrinsic and tensor derivatives obtained.

Moreover if we take special systems of space and surface coordinates in which the space coordinates are Cartesian and

the surface coordinates are geodesic at any given point of the surface, we immediately see that the intrinsic and tensor derivatives of a tensor reduce at the given point to ordinary derivatives of the tensor. Consequently the same rules apply to the intrinsic and tensor derivation of sums and products as apply to ordinary derivation. Also the tensor derivatives of the tensors g_{rs} , $a_{\alpha\beta}$, ε_{rst} , $\varepsilon_{\alpha\beta}$ and their associated tensors are all zero, since their ordinary derivatives all vanish in the special coordinate systems mentioned, so that they may be treated as constants in the process of tensor differentiation.

EXAMPLES.

1. Write down the formulae for the intrinsic and tensor derivatives of the tensor X^r_α .

2. If a tensor is an invariant for one system of coordinates (either space or surface), show that its tensor derivative is identical with the intrinsic or covariant derivative defined at pp. 147, 181.

6. Gauss's formulae. The second groundform of a surface.

Let us take the tensor x^r_α of § 2 (p. 194). Its tensor derivative is

$$x^r_{\alpha,\beta} = \frac{\partial^2 x^r}{\partial u^\alpha \partial u^\beta} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\}_g x^m_\alpha x^n_\beta - \left\{ \begin{matrix} \sigma \\ \alpha\beta \end{matrix} \right\}_a x^r_\sigma, \dots \quad (20)$$

which is symmetric in α, β .

We have the relation

$$g_{mn} x^m_\alpha x^n_\beta = a_{\alpha\beta}$$

and, if we take the tensor derivative of this equation, we get

$$g_{mn} x^m_{\alpha,\gamma} x^n_\beta + g_{mn} x^m_\alpha x^n_{\beta,\gamma} = a_{\alpha\beta,\gamma} = 0.$$

When we write down the two further equations obtained from this by interchanging α, β, γ cyclically, we find on adding two and subtracting one that

$$g_{mn} x^m_{\alpha,\beta} x^n_\gamma = 0, \dots \quad (21)$$

since $x^r_{\alpha,\beta}$ is symmetric in α, β . This result, interpreted geometrically, means that $x^r_{\alpha,\beta}$ from the point of view of the space-coordinates is a space vector orthogonal to the surface. It is therefore codirectional with the unit normal vector ξ^r and quantities $b_{\alpha\beta}$ must exist such that

$$x^r_{\alpha,\beta} = b_{\alpha\beta} \xi^r. \dots \quad (22)$$

We also conclude from this equation that $b_{\alpha\beta}$ is a symmetric double surface tensor and at the same time is a space invariant. The equations (22) are known as *Gauss's formulae*.

Moreover the differential quadratic form

$$B \equiv b_{\alpha\beta} du^\alpha du^\beta \dots \dots \dots (23)$$

is referred to as *the second groundform of the surface* and we denote it briefly by B .

EXAMPLES.

1. Prove that $b_{\alpha\beta} = g_{mn} x_{\alpha,\beta}^m \xi^n = \frac{1}{2} \varepsilon^{\sigma\tau} e_{mn\sigma} x_{\alpha,\beta}^m x_\sigma^n x_\tau^p$.
2. If the x -coordinates are Cartesian, show that

$$b_{\alpha\beta} = \frac{1}{2} \varepsilon^{\sigma\tau} e_{mn\sigma} \frac{\partial^2 x^m}{\partial u^\alpha \partial u^\beta} x_\sigma^n x_\tau^p.$$

3. The space coordinates are rectangular Cartesian and are chosen such that the tangent plane at the origin (which is on the surface) is $x^3 = 0$. If x^1 and x^2 are taken as the parameters u^1, u^2 , show that at the origin we have

$$(i) \quad x_1^r = \delta_1^r, \quad x_2^r = \delta_2^r; \quad (ii) \quad a_{11} = a_{22} = 1, \quad a_{12} = 0, \quad a = 1;$$

$$(iii) \quad \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\}_a = 0 = [\alpha\beta, \gamma]_a; \quad (iv) \quad \xi^r = (0, 0, 1);$$

$$(v) \quad \frac{\partial^2 x^1}{\partial u^\alpha \partial u^\beta} = \frac{\partial^2 x^2}{\partial u^\alpha \partial u^\beta} = 0, \quad \frac{\partial^2 x^3}{\partial u^\alpha \partial u^\beta} = b_{\alpha\beta}.$$

Deduce Gauss's formulae from the results (iv) and (v).

7. Weingarten's formulae. The third groundform of a surface.

We next consider the tensor derivative of the unit normal vector ξ^r . Since it is a contravariant space vector and a surface invariant we see that the tensor derivative is

$$\xi^r_{,\alpha} = \frac{\partial \xi^r}{\partial u^\alpha} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\}_g \xi^m x_\alpha^n \dots \dots \dots (24)$$

Now ξ^r is a unit vector-field and so

$$g_{mn} \xi^m \xi^n = 1.$$

Taking the tensor derivative of this equation we have

$$g_{mn} \xi^m_{,\alpha} \xi^n + g_{mn} \xi^m \xi^n_{,\alpha} = 0,$$

or

$$g_{mn} \xi^m_{,\alpha} \xi^n = 0.$$

This equation shows that the space vector $\xi^r_{,\alpha}$ is orthogonal to ξ^r and is therefore a tangent vector to the surface. Hence it can be expressed linearly in terms of the tangent vectors x^r_α . That is, there exist quantities η^β_α such that

$$\xi^r_{,\alpha} = \eta^\beta_\alpha x^r_\beta. \quad (25)$$

But since ξ^r is normal to the surface we have

$$g_{mn} x^m_\alpha \xi^n = 0.$$

Therefore, taking the tensor derivative of this equation,

$$g_{mn} x^m_{\alpha,\beta} \xi^n + g_{mn} x^m_\alpha \xi^n_{,\beta} = 0.$$

When we substitute from (22) and (25), this reads

$$b_{\alpha\beta} + g_{mn} x^m_\alpha x^n_\gamma \eta^\gamma_\beta = 0,$$

or

$$b_{\alpha\beta} = -a_{\alpha\gamma} \eta^\gamma_\beta.$$

Solving for η^γ_β we get

$$\eta^\gamma_\beta = -a^{\alpha\gamma} b_{\alpha\beta}.$$

Consequently (25) becomes

$$\xi^r_{,\alpha} = -a^{\beta\gamma} b_{\beta\alpha} x^r_\gamma, \quad (26)$$

which equations are known as *Weingarten's formulae*. They give expressions for the derivatives of the unit normal vector.

Having found expressions for the derivatives of ξ^r , we can form a third quadratic differential form. If we put

$$c_{\alpha\beta} = g_{mn} \xi^m_{,\alpha} \xi^n_{,\beta} \quad (27)$$

we see that $c_{\alpha\beta}$ is a symmetric double surface tensor, and we call the quadratic form

$$C \equiv c_{\alpha\beta} du^\alpha du^\beta \quad (28)$$

the *third groundform of the surface*. Using Weingarten's formulae in (27) we find that

$$c_{\alpha\beta} = g_{mn} a^{\sigma\mu} b_{\sigma\alpha} x^\mu_\mu a^{\tau\nu} b_{\tau\beta} x^\nu_\nu = a_{\mu\nu} a^{\sigma\mu} a^{\tau\nu} b_{\sigma\alpha} b_{\tau\beta},$$

or, as it may be written,

$$c_{\alpha\beta} = a^{\sigma\tau} b_{\sigma\alpha} b_{\tau\beta}. \quad (29)$$

EXAMPLES.

1. Prove that $g_{mn} \xi^m_{,\alpha} x^n_{,\beta} = -b_{\alpha\beta}$.

2. If $\beta^\gamma{}^\delta$ is the cofactor of $b_{\gamma\delta}$ in $|b_{\gamma\delta}|$, divided by $|b_{\gamma\delta}|$, show from (26) that

$$x^r_{,\alpha} = -a_{\alpha\delta} \beta^{\delta\gamma} \xi^r_{,\gamma}.$$

3. Prove that $a^{\alpha\beta} x^r_{,\alpha,\beta} = 2H\xi^r$, where we have put $H = \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta}$. H is called the mean curvature of the surface.

8. The equations of Gauss and Codazzi.

We have already considered the tensor derivative of $x^r_{,\alpha}$, namely, $x^r_{,\alpha,\beta}$. Let us take its tensor derivative or, what is the same thing, the second tensor derivative of $x^r_{,\alpha}$. It is

$$x^r_{,\alpha,\beta\gamma} = \frac{\partial x^r_{,\alpha,\beta}}{\partial u^\gamma} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\}_g x^m_{,\alpha,\beta} x^n_{,\gamma} - \left\{ \begin{matrix} \delta \\ \alpha\gamma \end{matrix} \right\}_a x^r_{,\delta,\beta} - \left\{ \begin{matrix} \delta \\ \beta\gamma \end{matrix} \right\}_a x^r_{,\alpha,\delta}. \quad (30)$$

Let us take for the moment the x -coordinates to be Cartesian and the u -coordinates to be geodesic at a given point P of the surface. Then the space Christoffel symbols are zero everywhere together with their derivatives whilst the surface Christoffel symbols all vanish at the point P . From (20), p. 200, and (30) we conclude that in such coordinate systems we have at P

$$x^r_{,\alpha,\beta\gamma} = \frac{\partial^3 x^r}{\partial u^\alpha \partial u^\beta \partial u^\gamma} - \frac{\partial}{\partial u^\gamma} \left\{ \begin{matrix} \sigma \\ \alpha\beta \end{matrix} \right\}_a x^r_{,\sigma},$$

all the other terms vanishing at this point. If we interchange β and γ in this formula and subtract the two results we have

$$x^r_{,\alpha,\beta\gamma} - x^r_{,\alpha,\gamma\beta} = \left[\frac{\partial}{\partial u^\beta} \left\{ \begin{matrix} \sigma \\ \alpha\gamma \end{matrix} \right\}_a - \frac{\partial}{\partial u^\gamma} \left\{ \begin{matrix} \sigma \\ \alpha\beta \end{matrix} \right\}_a \right] x^r_{,\sigma}. \quad (31)$$

But in our special coordinate systems (cf. p. 182)

$$R^{\sigma}_{\alpha\beta\gamma} = \frac{\partial}{\partial u^\beta} \left\{ \begin{matrix} \sigma \\ \alpha\gamma \end{matrix} \right\}_a - \frac{\partial}{\partial u^\gamma} \left\{ \begin{matrix} \sigma \\ \alpha\beta \end{matrix} \right\}_a$$

where $R^{\sigma}_{\alpha\beta\gamma}$ is the Riemann-Christoffel tensor of the surface. Hence (31) reads

$$x^r_{,\alpha,\beta\gamma} - x^r_{,\alpha,\gamma\beta} = R^{\sigma}_{\alpha\beta\gamma} x^r_{,\sigma}. \quad (32)$$

Now this is a tensor equation which is true in the special coordinates taken, and therefore (32) is true in all coordinates.

But (p. 200)

$$x_{\alpha,\beta}^r = b_{\alpha\beta} \xi^r.$$

Taking the tensor derivative of each side, we get by (26)

$$\begin{aligned} x_{\alpha,\beta\gamma}^r &= b_{\alpha\beta,\gamma} \xi^r + b_{\alpha\beta} \xi_{,\gamma}^r \\ &= b_{\alpha\beta,\gamma} \xi^r - b_{\alpha\beta} a^{\sigma\tau} b_{\sigma\gamma} x_{\tau}^r, \end{aligned}$$

where $b_{\alpha\beta,\gamma}$ is the tensor derivative of $b_{\alpha\beta}$, that is,

$$b_{\alpha\beta,\gamma} = \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \left\{ \begin{matrix} \sigma \\ \alpha\gamma \end{matrix} \right\}_a b_{\sigma\beta} - \left\{ \begin{matrix} \sigma \\ \beta\gamma \end{matrix} \right\}_a b_{\alpha\sigma}. \quad \dots (33)$$

Hence

$$x_{\alpha,\beta\gamma}^r - x_{\alpha,\gamma\beta}^r = (b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta}) \xi^r - a^{\sigma\tau} (b_{\alpha\beta} b_{\sigma\gamma} - b_{\alpha\gamma} b_{\sigma\beta}) x_{\tau}^r.$$

Substituting from (32) this becomes

$$R_{\alpha\beta\gamma}^{\sigma} x_{\sigma}^r = (b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta}) \xi^r - a^{\sigma\tau} (b_{\alpha\beta} b_{\sigma\gamma} - b_{\alpha\gamma} b_{\sigma\beta}) x_{\tau}^r. \quad (34)$$

If we first multiply (34) by ξ_r and use the fact that $x_{\alpha}^r \xi_r = 0$, we get

$$b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta} = 0. \quad \dots (35)$$

These equations are called *the Codazzi equations of the surface*.

Secondly, if we multiply (34) by $g_{rs} x_{\sigma}^s$ we obtain

$$R_{\alpha\beta\gamma} = b_{\alpha\beta} b_{\alpha\gamma} - b_{\alpha\gamma} b_{\alpha\beta}. \quad \dots (36)$$

These are called *the Gauss equations of the surface*. When we remember that Greek indices have the range of values 1 to 2 we see that there are really only *two* independent Codazzi equations and only *one* independent Gauss equation.

If we introduce the total curvature K of the surface by means of (52), p. 183, (36) may be written

$$K \varepsilon_{\alpha\beta} \varepsilon_{\beta\gamma} = b_{\alpha\beta} b_{\alpha\gamma} - b_{\alpha\gamma} b_{\alpha\beta}. \quad \dots (37)$$

When we multiply these equations by $a^{\alpha\gamma}$ and use the relations

$$a_{\alpha\beta} = -a^{\alpha\gamma} \varepsilon_{\alpha\gamma} \varepsilon_{\beta\gamma},$$

it follows that

$$-K a_{\alpha\beta} = a^{\alpha\gamma} b_{\alpha\beta} b_{\gamma\alpha} - a^{\alpha\gamma} b_{\alpha\gamma} b_{\alpha\beta}.$$

Now (p. 202)

$$a^{\alpha\gamma} b_{\alpha\beta} b_{\gamma\beta} = c_{\alpha\beta}.$$

Also we write

$$2H = a^{\epsilon\gamma} b_{\epsilon\gamma}, \quad (38)$$

and we call the invariant H the mean curvature of the surface.

Therefore

$$-K a_{\alpha\beta} = c_{\alpha\beta} - 2H b_{\alpha\beta},$$

or

$$c_{\alpha\beta} - 2H b_{\alpha\beta} + K a_{\alpha\beta} = 0. \quad (39)$$

From (39) we see that we have the following relation connecting the three groundforms of the surface

$$C - 2HB + KA = 0. \quad (40)$$

EXAMPLES.

$$1. \text{ Prove that } K = \frac{R_{1212}}{a} = \frac{b_{11}b_{22} - b_{12}^2}{a} = \frac{b}{a}.$$

$$2. \text{ Prove that } a^{\alpha\beta} c_{\alpha\beta} = 4H^2 - 2K.$$

3. If λ^α is a unit vector of the surface, show that

$$K = 2H b_{\alpha\beta} \lambda^\alpha \lambda^\beta - c_{\alpha\beta} \lambda^\alpha \lambda^\beta$$

$$[\text{In (40), } A = a_{\alpha\beta} du^\alpha dv^\beta = ds^2, \quad B = A b_{\alpha\beta} \frac{du^\alpha}{ds} \frac{dv^\beta}{ds},$$

$$C = A c_{\alpha\beta} \frac{du^\alpha}{ds} \frac{dv^\beta}{ds},$$

and we have only to put $\lambda^\alpha = \frac{du^\alpha}{ds}$.]

$$4. \text{ Prove that } g_{mn} \xi^m \xi^n_{,\alpha\beta} = -c_{\alpha\beta}.$$

[This result is obtained by taking the tensor derivative of the equation $g_{mn} \xi^m \xi^n_{,\alpha} = 0$.]

EXAMPLES XV:

1. If X^r is a unit vector, prove that

$$a^{\alpha\beta} x_\alpha^n x_\beta^n X_m X_n = \sin^2 \theta,$$

where θ is the angle between X^r and the unit normal vector ξ^r .

2. Prove that

$$\xi^r_{,\alpha\beta} = -c_{\alpha\beta} \xi^r - a^{\sigma\tau} b_{\sigma\alpha,\beta} x_\tau^r.$$

3. Prove that

$$a^{\alpha\beta} \xi^r_{,\alpha\beta} = -(4H^2 - 2K) \xi^r - 2a^{\alpha\beta} \frac{\partial H}{\partial u^\alpha} x_\beta^r.$$

[Here $\frac{\partial H}{\partial u^\alpha} = \frac{1}{2} a^{\sigma\tau} b_{\sigma\tau,\alpha}$ and use Codazzi's equations.]

4. If the equations of the surface are given in the form $x^1 = u^1$, $x^2 = u^2$, $x^3 = f(u^1, u^2)$ and the x -coordinates are rectangular Cartesian,

show that we have the following results, where we use the notation

$$p = \frac{\partial x^2}{\partial x^1}, \quad q = \frac{\partial x^3}{\partial x^2}, \quad r = \frac{\partial^2 x^3}{(\partial x^1)^2}, \quad s = \frac{\partial^2 x^3}{\partial x^1 \partial x^2}, \quad t = \frac{\partial^2 x^3}{(\partial x^2)^2} :$$

$$(i) \quad a_{11} = 1 + p^2, \quad a_{22} = 1 + q^2, \quad a_{12} = pq, \quad a = 1 + p^2 + q^2;$$

$$(ii) \quad a^{11} = \frac{1 + q^2}{1 + p^2 + q^2}, \quad a^{22} = \frac{1 + p^2}{1 + p^2 + q^2}, \quad a^{12} = -\frac{pq}{1 + p^2 + q^2};$$

$$(iii) \quad \xi^r = \left\{ -\frac{p}{\sqrt{1 + p^2 + q^2}}, \quad -\frac{q}{\sqrt{1 + p^2 + q^2}}, \quad \frac{1}{\sqrt{1 + p^2 + q^2}} \right\};$$

$$(iv) \quad b_{11} = \frac{r}{\sqrt{1 + p^2 + q^2}}, \quad b_{12} = \frac{s}{\sqrt{1 + p^2 + q^2}}, \quad b_{22} = \frac{t}{\sqrt{1 + p^2 + q^2}};$$

$$(v) \quad c_{11} = \frac{r^2 + s^2 + (ps - qr)^2}{(1 + p^2 + q^2)^2}, \quad c_{12} = \frac{st(1 + p^2) + rs(1 + q^2) - pq(rt + s^2)}{(1 + p^2 + q^2)^2},$$

$$c_{22} = \frac{s^2 + t^2 + (pt - qs)^2}{(1 + p^2 + q^2)^2};$$

$$(vi) \quad K = \frac{rt - s^2}{(1 + p^2 + q^2)^2}, \quad 2H = \frac{r(1 + q^2) - 2pq s + t(1 + p^2)}{(1 + p^2 + q^2)^{3/2}}.$$

5. If φ is a function of position on the surface, show that

$$\varphi^r = a^{\alpha\beta} \varphi_{,\alpha} x^r_{,\beta} = \frac{1}{a} x^r_1 \left(a_{22} \frac{\partial \varphi}{\partial u^1} - a_{12} \frac{\partial \varphi}{\partial u^2} \right) + \frac{1}{a} x^r_2 \left(a_{11} \frac{\partial \varphi}{\partial u^2} - a_{12} \frac{\partial \varphi}{\partial u^1} \right)$$

is a vector which is tangent to the surface.

6. Show that $\nabla \varphi = g_{mn} \varphi^m \varphi^n$, where φ^r is defined in Ex. 5.

7. If $\varphi = \text{const.}$, and $\psi = \text{const.}$ are taken as new parametric curves on the surface, φ, ψ being functions of u^α , show that the new metric tensor $\bar{a}_{\alpha\beta}$ is given (cf. p. 187) by

$$\bar{a}_{11} = \bar{a} \nabla \varphi, \quad \bar{a}_{12} = -\bar{a} \nabla (\varphi, \psi), \quad \bar{a}_{22} = \bar{a} \nabla \psi.$$

8. The surface divergence of a space vector. If A^r is a space vector defined over the surface, show that

$$g_{mn} a^{\alpha\beta} A^m_{,\alpha} x^a_{,\beta}$$

is an invariant. It is often referred to as the surface divergence of A^r .

9. Prove that the surface divergence of ξ^r equals $-2H$.

10. If we are given a family of surfaces $\varphi(x^1, x^2, x^3) = \text{const.}$, prove that the mean curvature of any member is given by

$$2H = -g^{rs} \left(\frac{\varphi_{,r}}{\sqrt{g^{mn} \varphi_{,m} \varphi_{,n}}} \right)_{,s}.$$

[Use the fact that $\xi_r = \varphi_{,r} / \sqrt{g^{mn} \varphi_{,m} \varphi_{,n}}$ where $\varphi_{,r} = \frac{\partial \varphi}{\partial x^r}$.]

11. If $A^r = C \xi^r + B^\alpha x_\alpha^r$, show that $C = A^r \xi_r$, $B^\alpha = a^{\alpha\beta} g_{mn} A^m x_\beta^n$, and prove that the surface divergence of A^r is given by

$$B^\alpha_{,\alpha} - 2 CH = \frac{1}{\sqrt{a}} \frac{\partial}{\partial u^\alpha} (\sqrt{a} B^\alpha) - 2 CH.$$

12. If we take a closed circuit C on the surface and integrate the surface divergence of A^r over its interior S , deduce from Ex. 11 that

$$\iint_S g_{mn} a^{\alpha\beta} A^\alpha_{,\alpha} x_\beta^n dS = \int_C B_\alpha \frac{du^\alpha}{ds} ds - 2 \iint_S CH dS.$$

13. *The surface curl of a space vector.* If A^r is a space vector defined over the surface, show that

$$\varepsilon_{rmn} a^{\alpha\beta} A^\alpha_{,\alpha} x_\beta^n$$

is a space vector. It is sometimes called *the surface curl of A^r* .

14. Prove that the surface curl of ξ^r is zero.

15. Prove that if we have a closed circuit C on the surface

$$\iint_S a^{\alpha\beta} \varepsilon_{rmn} \xi^r A^\alpha_{,\alpha} x_\beta^n dS = - \int_C g_{mn} A^m x_\alpha^n \frac{du^\alpha}{ds} ds. \quad [\text{ Cf. Ex. 3, p. 189. }]$$

CHAPTER XVI

CURVES ON A SURFACE

1. The equations of a curve on a surface.

As before we take the surface coordinates to be u^α and the space coordinates to be x^r . The equations of the surface are then of the form

$$x^r = x^r(u^1, u^2). \quad (1)$$

If the coordinates u^α are given as functions of a parameter then the point represented by these coordinates describes a curve on the surface as the parameter varies. We shall take as parameter the arc s of the curve and consequently the equations of a given curve of the surface are

$$u^\alpha = u^\alpha(s). \quad (2)$$

In consequence of (1) we see that the space coordinates of any point of the curve are also functions of s and the equations of the curve in space are

$$x^r = x^r(s). \quad (3)$$

Regarding the curve as a curve in space we shall denote its unit tangent vector by λ^r , its unit principal normal vector by μ^r and its unit binormal vector by ν^r . Then its curvature κ and its torsion τ are connected with these vectors by the Frenet formulae (p. 159)

$$\left. \begin{aligned} \frac{\partial \lambda^r}{\partial s} &= \kappa \mu^r, \\ \frac{\partial \mu^r}{\partial s} &= \tau \nu^r - \kappa \lambda^r, \\ \frac{\partial \nu^r}{\partial s} &= -\tau \mu^r. \end{aligned} \right\} \dots \dots \dots (4)$$

Regarding the curve as a curve on the surface we shall denote its unit tangent vector by λ^α and its unit normal vector in the surface by ϱ^α . Its geodesic curvature σ is connected with these vectors by the surface Frenet formulae (p. 185)

$$\left. \begin{aligned} \frac{\partial \lambda^\alpha}{\partial s} &= \sigma \varrho^\alpha, \\ \frac{\partial \varrho^\alpha}{\partial s} &= -\sigma \lambda^\alpha. \end{aligned} \right\} \dots \dots \dots (5)$$

EXAMPLES.

1. Prove that $\lambda^r = x_\alpha^r \lambda^\alpha$, where $x_\alpha^r = \frac{\partial x^r}{\partial u^\alpha}$.
2. Prove that the space vector perpendicular to λ^r and tangent to the surface is

$$\varrho^r = x_\alpha^r \varrho^\alpha.$$
3. If θ is the angle between μ^r and the unit surface normal ξ^r , prove that $\sin \theta = g_{mn} \mu^m x_\alpha^n \varrho^\alpha$.

2. Meusnier's theorem.

We know that

$$\lambda^r = \frac{dx^r}{ds}, \quad \lambda^\alpha = \frac{du^\alpha}{ds}, \quad \dots \dots \dots (6)$$

and x^r, u^α are related by means of the equations (1). Hence

$$\lambda^r = \frac{\partial x^r}{\partial u^\alpha} \frac{du^\alpha}{ds} = x_\alpha^r \lambda^\alpha.$$

If we take the intrinsic derivative of this equation with respect to s we have

$$\frac{d\lambda^r}{ds} = x_\alpha^r \frac{d\lambda^\alpha}{ds} + x_{\alpha,\beta}^r \lambda^\alpha \frac{du^\beta}{ds}.$$

From the Frenet formulae (4) and (5) this becomes

$$\kappa \mu^r = \sigma x'_\alpha \varrho^\alpha + x'_{\alpha\beta} \lambda^\alpha \lambda^\beta.$$

Let us denote by ϱ^r the space vector giving the same direction on the surface as ϱ^α , that is, $\varrho^r = x'_\alpha \varrho^\alpha$. Using (22), p. 200, we have

$$\kappa \mu^r = \sigma \varrho^r + b_{\alpha\beta} \lambda^\alpha \lambda^\beta \xi^r, \quad \dots \dots \dots (7)$$

where ξ^r is the unit surface normal.

Let θ be the angle between the principal normal μ^r and the surface normal ξ^r . Then

$$\cos \theta = \xi_r \mu^r \quad \dots \dots \dots (8)$$

and, when we multiply (7) by ξ_r , we get

$$\kappa \cos \theta = b_{\alpha\beta} \lambda^\alpha \lambda^\beta. \quad \dots \dots \dots (9)$$

The quantity $b_{\alpha\beta} \lambda^\alpha \lambda^\beta$ is the same for all curves of the surface which have the same tangent vector λ^α and consequently $\kappa \cos \theta$ is also the same for all such curves. We have therefore *Meusnier's theorem*:

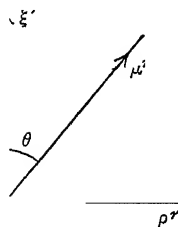
For all curves on a surface which have the same tangent vector, the quantity $\kappa \cos \theta$ has the same value.

The quantity $\kappa \cos \theta$ is called the normal curvature of the surface in the direction λ^α and we may denote it by $\kappa_{(n)}$. Hence

$$\kappa_{(n)} = b_{\alpha\beta} \lambda^\alpha \lambda^\beta, \quad (10)$$

which is the formula giving the normal curvature in any direction.

In particular, if we take a plane section of the surface which is normal to it, we see that in such a case θ is either 0 or π and hence $\kappa \cos \theta$ is either $+\kappa$ or $-\kappa$. In other words the normal curvature of a surface in any direction is equal *in magnitude* to the curvature of the normal plane section of the surface in that direction. This is the reason that the name "normal curvature" is applied to the quantity $\kappa \cos \theta$.



Moreover since ϱ^r is perpendicular to λ^r it lies in the plane containing ξ^r and μ^r . Also it is tangent to the surface and hence the angle between μ^r and ϱ^r is $\left(\frac{\pi}{2} - \theta\right)$. If we multiply

the formula (7) by q_r we therefore get

$$\kappa \cos \left(\frac{\pi}{2} - \theta \right) = \sigma,$$

or

$$\sigma = \kappa \sin \theta,$$

which is the formula connecting the geodesic curvature of a surface curve with its curvature.

EXAMPLES.

1. Prove that $\kappa_{(n)} = \frac{b_{\alpha\beta} du^\alpha dv^\beta}{a_{\alpha\beta} du^\alpha dv^\beta}$.
2. Prove that the normal curvatures in the directions of the coordinate curves are b_{11}/a_{11} and b_{22}/a_{22} .
3. *If a curve is a geodesic on the surface, prove that it is either a straight line or its principal normal is orthogonal to the surface at every point, and conversely.*
[For a geodesic $\sigma = 0$ and therefore either $\kappa = 0$, or $\theta = 0$ or π .]

3. The principal curvatures. Gauss's theorem.

The normal curvature of a surface in the direction λ^α is

$$\kappa_{(n)} = b_{\alpha\beta} \lambda^\alpha \lambda^\beta,$$

where λ^α is subject to the relation

$$a_{\alpha\beta} \lambda^\alpha \lambda^\beta = 1.$$

Consequently the directions for which the normal curvature is $\kappa_{(n)}$ are given by the quadratic equation

$$(b_{\alpha\beta} - \kappa_{(n)} a_{\alpha\beta}) \lambda^\alpha \lambda^\beta = 0. \quad (11)$$

To find the maximum and minimum values of $\kappa_{(n)}$, the usual method tells us that a direction giving a maximum or minimum value of $\kappa_{(n)}$ satisfies

$$(b_{\alpha\beta} - \kappa_{(n)} a_{\alpha\beta}) \lambda^\beta = 0, \quad (12)$$

and therefore the corresponding value of $\kappa_{(n)}$ must be a root of the equation in θ

$$|b_{\alpha\beta} - \theta a_{\alpha\beta}| = 0. \quad (13)$$

This equation has two roots and these roots are the maximum and minimum values of the normal curvature.

When (13) is written out in full it becomes

$$\theta^2 - a^{\alpha\beta} b_{\alpha\beta} \theta + b/a = 0$$

or, substituting the values of H and K (p. 205)

$$\theta^2 - 2H\theta + K = 0. \quad (14)$$

The roots of this equation are called *the principal curvatures of the surface* and we shall denote them by κ_1 and κ_2 . Thus

$$\left. \begin{aligned} \kappa_1 + \kappa_2 &= 2H \\ \kappa_1 \kappa_2 &= K. \end{aligned} \right\} \quad (15)$$

Hence $2H$ is the sum of the principal curvatures of the surface and K , the Gaussian curvature, is the product of the principal curvatures.

We have seen in the chapter on the intrinsic geometry of a surface that K is intrinsic to the surface and depends only on the $a_{\alpha\beta}$. This remark gives *Gauss's theorem*:

The product of the principal curvatures of a surface is an intrinsic invariant of the surface, namely, the Gaussian curvature.

EXAMPLES.

1. Find the equation for the principal curvatures of the surface $x^1 = u^1$, $x^2 = u^2$, $x^3 = f(u^1, u^2)$, the x -coordinates being rectangular Cartesian.

2. Prove that, for the surface $x^r = f^r(u^1) + u^2 \frac{df^r}{du^1}$, we have $\kappa_1 \kappa_2 = 0$.

[This is the surface generated by the tangents to the space curve $x^r = f^r(u^1)$.]

4. The lines of curvature.

We shall now consider *the directions* which give the principal curvatures. These are called *the principal directions* and there are obviously two at every point. The principal directions are given by putting κ_1 and κ_2 for $\kappa_{(u)}$ in (12), and hence if we denote the principal directions by $\lambda_{(1)}^\alpha$ and $\lambda_{(2)}^\alpha$ we have

$$\left. \begin{aligned} (b_{\alpha\beta} - \kappa_1 a_{\alpha\beta}) \lambda_{(1)}^\beta &= 0, \\ (b_{\alpha\beta} - \kappa_2 a_{\alpha\beta}) \lambda_{(2)}^\beta &= 0. \end{aligned} \right\} \quad (16)$$

If we multiply the first of these equations by $\lambda_{(2)}^\alpha$, and the second by $\lambda_{(1)}^\alpha$, and then subtract we get

$$(\kappa_2 - \kappa_1) a_{\alpha\beta} \lambda_{(1)}^\alpha \lambda_{(2)}^\beta = 0.$$

We shall assume that $\kappa_2 \neq \kappa_1$, and consequently

$$a_{\alpha\beta} \lambda_{(1)}^\alpha \lambda_{(2)}^\beta = 0,$$

which shows that *the principal directions are orthogonal*. Also

$$b_{\alpha\beta} \lambda_{(1)}^\alpha \lambda_{(2)}^\beta = \kappa_1 a_{\alpha\beta} \lambda_{(1)}^\alpha \lambda_{(2)}^\beta = 0. \quad . \quad . \quad . \quad (17)$$

If we take any one of the two sets of equations (16), say the first, we can write it

$$\begin{aligned} b_{1\beta} \lambda_{(1)}^\beta &= \kappa_1 a_{1\alpha} \lambda_{(1)}^\alpha, \\ b_{2\beta} \lambda_{(1)}^\beta &= \kappa_1 a_{2\alpha} \lambda_{(1)}^\alpha. \end{aligned}$$

Eliminating κ_1 between these two equations, we obtain

$$a_{1\alpha} \lambda_{(1)}^\alpha b_{2\beta} \lambda_{(1)}^\beta = a_{2\alpha} \lambda_{(1)}^\alpha b_{1\beta} \lambda_{(1)}^\beta.$$

In other words, the direction $\lambda_{(1)}^\alpha$ satisfies the equation

$$\varepsilon^{\gamma\delta} a_{\alpha\gamma} b_{\beta\delta} \lambda^\alpha \lambda^\beta = 0. \quad . \quad . \quad . \quad . \quad (18)$$

Similarly $\lambda_{(2)}^\alpha$ satisfies this equation, and (18) is the combined equation of the principal directions. If we write

$$h_{\alpha\beta} = \varepsilon^{\gamma\delta} a_{\alpha\gamma} b_{\beta\delta}, \quad . \quad . \quad . \quad . \quad (19)$$

the equation for the principal directions takes the form

$$h_{\alpha\beta} \lambda^\alpha \lambda^\beta = 0. \quad . \quad . \quad . \quad . \quad (20)$$

A curve on the surface which is tangent at each of its points to one of the principal directions at that point is called a *line of curvature*. Thus if du^α is a displacement along a line of curvature, then du^α will be proportional either to $\lambda_{(1)}^\alpha$ or $\lambda_{(2)}^\alpha$ and

$$h_{\alpha\beta} du^\alpha du^\beta = 0, \quad . \quad . \quad . \quad . \quad (21)$$

which is the equation of the lines of curvature on the surface.

EXAMPLES.

$$1. \text{ Prove that } \kappa_1 = b_{\alpha\beta} \lambda_{(1)}^\alpha \lambda_{(1)}^\beta, \quad \kappa_2 = b_{\alpha\beta} \lambda_{(2)}^\alpha \lambda_{(2)}^\beta.$$

2. *Euler's theorem*: If the direction λ^α makes an angle θ with the principal direction $\lambda_{(1)}^\alpha$, show that the normal curvature for that direction is

$$\kappa_{(n)} = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta.$$

[We have $\lambda^\alpha = \lambda_{(1)}^\alpha \cos \theta + \lambda_{(2)}^\alpha \sin \theta$. Then $\kappa_{(n)} = b_{\alpha\beta} \lambda^\alpha \lambda^\beta$ and the result follows from (17) and Ex. 1.]

3. Prove that

$$h_{\alpha\beta} \lambda_{(1)}^{\alpha} \lambda_{(2)}^{\beta} = \kappa_2, \quad h_{\alpha\beta} \lambda_{(2)}^{\alpha} \lambda_{(1)}^{\beta} = -\kappa_1.$$

[We have $h_{\alpha\beta} \lambda_{(1)}^{\alpha} \lambda_{(2)}^{\beta} = \varepsilon^{\gamma\delta} a_{\gamma\alpha} b_{\delta\beta} \lambda_{(1)}^{\alpha} \lambda_{(2)}^{\beta} = \kappa_2 \varepsilon^{\gamma\delta} a_{\gamma\alpha} a_{\delta\beta} \lambda_{(1)}^{\alpha} \lambda_{(2)}^{\beta}$
 $= \kappa_2 \varepsilon_{\alpha\beta} \lambda_{(1)}^{\alpha} \lambda_{(2)}^{\beta} = \kappa_2$, (Ex. 2, p. 170)].

4. If λ^{α} is a direction making an angle θ with a line of curvature, show that

$$h_{\alpha\beta} \lambda^{\alpha} \lambda^{\beta} = (\kappa_2 - \kappa_1) \sin \theta \cos \theta.$$

[$\lambda^{\alpha} = \lambda_{(1)}^{\alpha} \cos \theta + \lambda_{(2)}^{\alpha} \sin \theta$ and use Ex. 3.]

5. If the u -curves are taken as the lines of curvature on the surface show that $a_{12} = b_{12} = 0$.

5. The asymptotic lines. Enneper's formula.

The directions on the surface given by the equations

$$b_{\alpha\beta} \lambda^{\alpha} \lambda^{\beta} = 0 \quad . \quad . \quad . \quad . \quad . \quad (22)$$

are called *the asymptotic directions*, and the curves whose tangents are asymptotic directions at each point are called *the asymptotic lines of the surface*. There are obviously two asymptotic directions at every point and hence there are two asymptotic lines passing through each point of the surface.

If we refer back to equation (7), p. 209, we see that for an asymptotic line

$$\kappa \mu^r = \sigma \varrho^r.$$

Consequently

$$\kappa = \pm \sigma, \quad \mu^r = \pm \varrho^r, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and we conclude that *the curvature and geodesic curvature of an asymptotic line are equal in magnitude*. Also its principal normal is tangent to the surface and hence *the binormal of an asymptotic line coincides with the surface normal*.

We shall now find a well-known formula for the torsion of an asymptotic line. We have for an asymptotic line

$$\nu^r = \pm \xi^r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

When we take the intrinsic derivative of this equation we get (p. 208)

$$-\tau \mu^r = \frac{\delta \nu^r}{\delta s} = \pm \frac{\delta \xi^r}{\delta s} = \pm \xi^r_{,\alpha} \lambda^{\alpha}.$$

Consequently (p. 202)

$$\tau^2 = g_{mn} (\tau \mu^m) (\tau \mu^n) = g_{mn} \xi^m_{,\alpha} \xi^n_{,\beta} \lambda^{\alpha} \lambda^{\beta} = c_{\alpha\beta} \lambda^{\alpha} \lambda^{\beta}.$$

But (p. 205)

$$c_{\alpha\beta} - 2Hb_{\alpha\beta} + Ka_{\alpha\beta} = 0,$$

and hence if λ^α is an asymptotic direction

$$c_{\alpha\beta} \lambda^\alpha \lambda^\beta = -Ka_{\alpha\beta} \lambda^\alpha \lambda^\beta = -K.$$

We conclude that

$$\tau = \pm \sqrt{-K}, \dots \dots \dots (25)$$

which proves *Enneper's formula*: *The torsion of an asymptotic line equals $\pm \sqrt{-K}$, where K is the total curvature of the surface.*

EXAMPLES.

1. Prove that for an asymptotic direction the normal curvature is zero.

2. If the u -curves on the surface are taken to be the asymptotic lines, prove that $b_{11} = b_{22} = 0$ and show that

$$K = \frac{c_{12}}{a}, \quad H = \frac{a_{12}b_1}{a}, \quad \frac{a_{12}}{b_{12}} = \frac{H}{K}.$$

3. If ψ is the angle between the asymptotic directions prove that

$$\tan \frac{\psi}{2} = \sqrt{-\frac{\kappa_1}{\kappa_2}}.$$

6. The geodesic torsion of a curve on a surface.

Let θ be the angle between ξ^r and μ^r , θ being measured positively from μ^r to ν^r . Then

$$\cos \theta = \xi^r \mu_r.$$

Differentiating this equation we have

$$\begin{aligned} -\sin \theta \frac{d\theta}{ds} &= \frac{\delta \xi^r}{\delta s} \mu_r + \xi^r \frac{\delta \mu_r}{\delta s} \\ &= \frac{\delta \xi^r}{\delta s} \mu_r + \xi^r (\tau \nu_r - \kappa \lambda_r), \end{aligned}$$

by the Frenet formulae. Hence, since $\xi^r \nu_r = \sin \theta$,

$$\begin{aligned} -\sin \theta \left(\tau + \frac{d\theta}{ds} \right) &= \frac{\delta \xi^r}{\delta s} \mu_r = \xi^r_{,a} \lambda^a \mu_r \\ &= -a^{\beta\gamma} b_{\beta\alpha} \lambda^\alpha x^\alpha_\gamma \mu_r, \end{aligned}$$

where we have used Weingarten's formulae (p. 202).

Also, referring to the figure on p. 209,

$$\mu^r = \xi^r \cos \theta + \varrho^r \sin \theta,$$

that is,

$$\begin{aligned} \sin \left(\tau + \frac{d\theta}{ds} \right) &= -\sin \theta a^{\beta\gamma} b_{\beta\alpha} \lambda^\alpha x_\gamma^r \varrho_r \\ &= -\sin \theta a^{\beta\gamma} b_{\beta\alpha} \lambda^\alpha \varrho_\gamma = -\sin \theta b_{\beta\alpha} \varrho^\beta \lambda^\alpha. \end{aligned}$$

But

$$\varrho^\beta = -\varepsilon^{\beta\gamma} \lambda_\gamma,$$

and therefore

$$\tau + \frac{d\theta}{ds} = -\varepsilon^{\beta\gamma} b_{\beta\alpha} \lambda^\alpha \lambda_\gamma = \varepsilon^{\gamma\delta} a_{\gamma\alpha} b_{\delta\beta} \lambda^\alpha \lambda^\beta = h_{\alpha\beta} \lambda^\alpha \lambda^\beta.$$

We conclude from this equation that the quantity $\left(\tau + \frac{d\theta}{ds} \right)$ is the same for all curves of the surface which have the same tangent vector λ^α . In particular, if we take the geodesic in this direction θ is always 0 or π and $\left(\tau + \frac{d\theta}{ds} \right)$ equals the torsion of the geodesic. Now we define the *geodesic torsion of a curve on the surface at any point on it as the torsion of the geodesic which touches the curve at the point*. If we denote the geodesic torsion by τ_g we have therefore the formula

$$\tau_g = \tau + \frac{d\theta}{ds} = h_{\alpha\beta} \lambda^\alpha \lambda^\beta. \quad \dots \dots \dots (26)$$

EXAMPLES.

1. If the direction λ^α makes an angle θ with one of the lines of curvature, prove that

$$\tau_g = (\kappa_2 - \kappa_1) \sin \theta \cos \theta.$$

2. Deduce from Ex. 1 that the geodesic torsion of a curve on the surface is zero only when the curve is a line of curvature.

EXAMPLES XVI.

1. Prove that for any curve on a surface $\kappa^2 = \sigma^2 + \kappa_{(n)}^2$.

2. If a geodesic on a surface is a plane curve, show that it is also a line of curvature and conversely.

[Its geodesic torsion is zero.]

3. If a curve is the intersection of two surfaces, which cut at a constant angle, show that the geodesic torsions of the curve on the two surfaces have the same value.

[Both τ and $\frac{d\theta}{ds}$ have the same value for the curve.]

4. Deduce *Joachimsthal's theorem*: If a curve is the intersection of two surfaces, which cut at a constant angle, prove that if it is a line of curvature on one surface it is also a line of curvature on the other.

5. If two curves on a surface cut at right angles, show that the sum of their geodesic torsions is zero.

[Use Ex. 1, p. 215.]

6. Deduce *Dupin's theorem*: If three surfaces cut mutually at right angles, the curves of intersection are lines of curvature on each surface.

7. If we take the derivative of ξ^r along a curve on the surface, prove that

$$\frac{\delta \xi^r}{\delta s} = -a^{\beta\gamma} b_{\beta\alpha} x_\gamma^r \lambda^\alpha.$$

[This follows from Weingarten's formulae.]

8. *Rodrigues' formulae*. Prove that along a line of curvature on a surface

$$\frac{\delta \xi^r}{\delta s} + \kappa \frac{dx^r}{ds} = 0,$$

where κ is the appropriate principal curvature of the surface.

[This follows from Ex. 7 when we use the equation $b_{\alpha\beta} \lambda^\beta = \kappa a_{\alpha\beta} \lambda^\beta$, which is true for a principal direction.]

9. Prove the converse of Ex. 8.

10. Two directions $\lambda^\alpha, \mu^\alpha$ of the surface are said to be *conjugate* if they satisfy the equation

$$b_{\alpha\beta} \lambda^\alpha \mu^\beta = 0,$$

and two systems of curves on the surface whose tangents at any point are conjugate directions are said to be *conjugate systems*.

(i) If the coordinate curves are conjugate prove that $b_{12} = 0$.

(ii) Prove that the necessary and sufficient condition that the directions given by $p_{\alpha\beta} \lambda^\alpha \lambda^\beta = 0$ should be conjugate is $\beta^{\alpha\beta} p_{\alpha\beta} = 0$, where $\beta^{\alpha\beta}$ is the cofactor of $b_{\alpha\beta}$ in $|b_{\alpha\beta}|$, divided by $|b_{\alpha\beta}|$.

(iii) Show that the curves $\varphi = \text{const.}$, $\psi = \text{const.}$ form a conjugate system if $\beta^{\alpha\beta} \frac{\partial \varphi}{\partial u^\alpha} \frac{\partial \psi}{\partial u^\beta} = 0$.

(iv) If $\lambda^\alpha, \mu^\alpha$ are two conjugate directions making an angle φ with each other, prove that $c_{\alpha\beta} \lambda^\alpha \mu^\beta = -K \cos \varphi$.

(v) If two conjugate directions make angles θ, θ' with a principal direction, show that $\tan \theta \tan \theta' = -\kappa_1/\kappa_2$.

11. *Umbilics*. At a point on a surface where the principal curvatures are equal, prove that every direction is a principal direction and the normal curvature in every direction is the same. Such a point is called an *umbilic*. Show that the condition for an umbilic is

$$b_{\alpha\beta} = \kappa a_{\alpha\beta}.$$

12. *Minimal Surfaces*. A *minimal surface* is one for which $H = 0$ at every point. Show that for a minimal surface the asymptotic lines form an orthogonal system.

13. Prove that $\tau_g^2 = c_{\alpha\beta} \lambda^\alpha \lambda^\beta - \kappa_{(n)}^2$.

14. If κ and τ are the curvature and torsion of a geodesic, which makes an angle φ with a principal direction, prove that

$$(i) \begin{cases} \kappa \cos \varphi - \tau \sin \varphi = \kappa_1 \cos \varphi \\ \kappa \sin \varphi + \tau \cos \varphi = \kappa_2 \sin \varphi \end{cases}; \quad (ii) \tau^2 = -(\kappa - \kappa_1)(\kappa - \kappa_2).$$

15. Show that the geodesic torsion of a curve, whose tangent is an asymptotic direction, equals $\pm \sqrt{-K}$. Deduce Enneper's formula for the torsion of an asymptotic line.

16. *Laguerre's formula*. By taking the intrinsic derivative of equation (9), p. 209, with respect to s , prove that

$$L \equiv \frac{d\kappa}{ds} \cos \theta - \kappa \sin \theta \left(2\tau + 3 \frac{d\theta}{ds} \right) = b_{\alpha\beta, \gamma} \lambda^\alpha \lambda^\beta \lambda^\gamma.$$

This result shows that L has the same value for all curves on the surface which have the same tangent vector.

17. Using the fact that τ_g and L are the same for any two curves in the same direction, prove that the curvature at P of one of the curves in which the tangent plane at P cuts the surface equals *two-thirds* of the curvature of the corresponding asymptotic line.

At the point P , $\theta = \frac{\pi}{2}$. Also for an asymptotic line, $\theta = \frac{\pi}{2}$, and therefore $\tau_g = \tau$, $L = -2\kappa\tau$. For the plane curve, denoting corresponding quantities by dashes, we have $\tau' \neq 0$ and therefore $\tau_g = \frac{d\theta'}{ds}$, $L = -3\kappa' \frac{d\theta'}{ds}$. Equating the two values of τ_g and L we obtain our result.]

PART IV

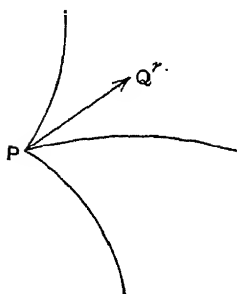
Applied Mathematics

CHAPTER XVII

DYNAMICS OF A PARTICLE

1. The equations of motion.

Let us suppose that we have a particle P moving in space and let us suppose that its position is given by a system of curvilinear coordinates, which we shall denote by x^r .*



As the time t varies P will describe a certain curve in space, called *the trajectory of the particle*, and the equations of this curve will generally be given by expressing the coordinates of P as functions of the time, that is,

$$x^r = x^r(t). \quad \dots \quad (1)$$

If we transform to another curvilinear coordinate system \bar{x}^r in accordance with the equations of transformation

$$\bar{x}^r = f^r(x^1, x^2, x^3), \quad \dots \quad (2)$$

the coordinates of P in the new system are given in terms of t by substituting (1) in (2).

Let us consider the quantities

$$v^r = \frac{dx^r}{dt} \quad \dots \quad (3)$$

In the new coordinates \bar{x}^r the corresponding quantities are

$$\bar{v}^r = \frac{d\bar{x}^r}{dt} = \frac{\partial \bar{x}^r}{\partial x^s} \frac{dx^s}{dt} = \frac{\partial \bar{x}^r}{\partial x^s} v^s.$$

* We refer the student to Chapters XI and XII for the geometrical properties of curvilinear coordinates in space.

and these equations express the fact that the quantities (3) are the components of a *contravariant vector*. If the coordinates are chosen to be rectangular Cartesian then (3) become the components of the velocity of P along the coordinate axes. Consequently we call (3) *the velocity vector of the particle*.

Again the vector v^r is itself a function of the time and we may take its intrinsic derivative with respect to t . This gives the vector (cf. p. 145)

$$f^r \equiv \frac{dv^r}{dt} \equiv \frac{d^2 x^r}{dt^2} + \left\{ \begin{matrix} r \\ mn \end{matrix} \right\} \frac{dx^m}{dt} \frac{dx^n}{dt}, \quad \dots \quad (4)$$

where $\left\{ \begin{matrix} r \\ mn \end{matrix} \right\}$ are the Christoffel symbols of our coordinate system.

When the coordinates are rectangular Cartesian the Christoffel symbols are all identically zero and the vector f^r becomes

$$f^r = \frac{d^2 x^r}{dt^2},$$

which gives the components of acceleration along the three coordinate axes. We therefore call (4) *the acceleration vector of the particle*.

The *mass* of a particle is obviously a quantity quite independent of the coordinate system used and is consequently an *invariant*. We shall denote it by M .

If a free particle be acted upon by a force, Newton's Law of Motion states that the force is in the direction of the acceleration and its magnitude (in appropriate units) is equal to the product of the mass and the acceleration. Hence, if we consider the vector equation

$$Q^r = M f^r, \quad \dots \quad (5)$$

where Q^r is a contravariant vector, we see that the quantities Q^r measure the force completely in magnitude and direction. It is therefore called *the force vector* and the equations (5) are the equations of motion in curvilinear coordinates.

In Cartesian coordinates, equations (5) take the well-known form

$$Q^r = M \frac{d^2 x^r}{dt^2}$$

and Q^1, Q^2, Q^3 are the components of the force along the three axes of coordinates.

EXAMPLES.

1. If X^1, X^2, X^3 are the components of a force in a Cartesian system of axes (y^1, y^2, y^3) , show that the force components in any curvilinear system x^r are given by the equations

$$\lambda^r = X^s \frac{\partial x^r}{\partial y^s}.$$

[These equations merely express the laws of transformation of a contravariant vector whose components are X^r in the coordinates y^r .]

2. If the line-element of our coordinate system is given by $ds^2 = g_{mn} dx^m dx^n$, show that

(i) the magnitude of the velocity is $(g_{mn} v^m v^n)^{\frac{1}{2}}$;

(ii) the magnitude of the acceleration is $(g_{mn} f^m f^n)^{\frac{1}{2}}$.

3. If a particle is moving with uniform velocity show that $f^r = 0$.

4. If v is the magnitude of the velocity, show that

$$\frac{dv}{dt} = f \cos \theta,$$

where f is the magnitude of the acceleration and θ is the angle between the velocity and acceleration vectors. Deduce that, if the velocity of a particle is constant in magnitude, then the acceleration must be either zero or orthogonal to the velocity vector.

[We have $v^2 = g_{mn} v^m v^n$, and hence $2v \frac{dv}{dt} = 2g_{mn} v^m \frac{\delta v^n}{\delta t} = 2g_{mn} v^m f^n$.

Also $g_{mn} v^m f^n = v f \cos \theta$ and the result follows immediately.]

2. Work and energy. Lagrange's equations of motion.

In the last section we have introduced *force* as a *contravariant* vector, but it can also be introduced as a *covariant* vector by means of the idea of work.

We know that, if the axes \bar{x}^r are rectangular Cartesian, a force whose components in these coordinates are \bar{Q}^r and whose point of application is moved through a small displacement $\delta \bar{x}^r$ does an amount of work measured by

$$\delta W = \bar{Q}^1 \delta \bar{x}^1 + \bar{Q}^2 \delta \bar{x}^2 + \bar{Q}^3 \delta \bar{x}^3.$$

Now in orthogonal Cartesian coordinates the associated vector \bar{Q}_r has exactly the same components as \bar{Q}^r and the expression for δW becomes

$$\delta W = \bar{Q}_r \delta \bar{x}^r.$$

From the relation connecting \bar{x}^r and x^r we see that δW can be written as a linear function of δx^r , thus

$$\delta W = Q_r \delta x^r. \quad (6)$$

Also δW is an invariant and we conclude from (6) that the coefficients of the linear form, Q_r , form a covariant vector. Moreover, since its components in an orthogonal Cartesian coordinate system are those of the force vector, it follows that Q_r is the covariant force vector in the coordinates x^r . The covariant and contravariant components are of course connected by the formulae

$$Q_r = g_{rs} Q^s, \quad Q^r = g^{rs} Q_s.$$

If $Q_r dx^r$ is a perfect differential, then the force is said to be conservative and we may write

$$W = \int Q_r dx^r. \quad (7)$$

The function W , which is determined by (7) except for an arbitrary constant, is called the *work function*. It is more usual to use the function V , called the *force potential*, which equals $-W$. From (7) it follows that

$$Q_r = -\frac{\partial V}{\partial x^r}. \quad (8)$$

We shall next consider the *kinetic energy* of a particle. It is defined as $\frac{1}{2} M v^2$, where v is the magnitude of the velocity. But

$$v^2 = g_{mn} v^m v^n,$$

where g_{mn} is the metric tensor of our curvilinear coordinates. Consequently the kinetic energy T is

$$T = \frac{1}{2} M g_{mn} v^m v^n = \frac{1}{2} M g_{mn} \dot{x}^m \dot{x}^n, \quad (9)$$

where $\dot{x}^r = \frac{dx^r}{dt}$.

If we differentiate T partially with respect to \dot{x}^r , we have

$$\frac{\partial T}{\partial \dot{x}^r} = M g_{rm} \dot{x}^m.$$

Hence

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}^r} \right) = M \left(g_{rm} \ddot{x}^m + \frac{\partial g_{rm}}{\partial x^n} \dot{x}^m \dot{x}^n \right).$$

Also, differentiating T partially with respect to x^r ,

$$\frac{\partial T}{\partial x^r} = \frac{1}{2} M \frac{\partial g_{mn}}{\partial x^r} \dot{x}^m \dot{x}^n.$$

From these equations we deduce that

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^r}\right) - \frac{\partial T}{\partial x^r} &= M\left\{g_{rm}\ddot{x}^m + \frac{1}{2}\left(\frac{\partial g_{rm}}{\partial x^n} + \frac{\partial g_{rn}}{\partial x^m} - \frac{\partial g_{mn}}{\partial x^r}\right)\dot{x}^m\dot{x}^n\right\} \\ &= M(g_{rm}\ddot{x}^m + [mn, r]\dot{x}^m\dot{x}^n),\end{aligned}$$

$[mn, r]$ being the first Christoffel symbol. Therefore

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^r}\right) - \frac{\partial T}{\partial x^r} = M g_{rs}\left(\ddot{x}^s + \left\{\begin{smallmatrix} s \\ mn \end{smallmatrix}\right\}\dot{x}^m\dot{x}^n\right)$$

Referring to (4) we see that the right-hand side equals $M g_{rs} f^s$. Denoting the covariant components of the acceleration vector by f_r , it follows that

$$M f_r = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^r}\right) - \frac{\partial T}{\partial x^r} \dots \dots \dots (10)$$

This formula is the most convenient to use when we wish to calculate the acceleration vector in any special coordinate system.

The equations of motion (5), p. 219, can now be written in the covariant form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^r}\right) - \frac{\partial T}{\partial x^r} = Q_r, \dots \dots \dots (11)$$

which is known as Lagrange's form of the equations. If the force system is conservative, these equations read

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^r}\right) - \frac{\partial T}{\partial x^r} = -\frac{\partial V}{\partial x^r},$$

or, what is equivalent,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^r}\right) - \frac{\partial L}{\partial x^r} = 0, \dots \dots \dots (12)$$

where

$$L = T - V. \dots \dots \dots (13)$$

The function L is called *the Lagrangian function*.

EXAMPLES.

1. Find the covariant components of the acceleration vector in spherical polar coordinates.

In spherical polars the line-element is (p. 136)

$$ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1)^2(\sin x^2)^2(dx^3)^2.$$

Hence the kinetic energy is given by

$$T = \frac{1}{2} M \left(\frac{ds}{dt}\right)^2 = \frac{1}{2} M \{(\dot{x}^1)^2 + (x^1 \dot{x}^2)^2 + (x^1 \sin x^2 \dot{x}^3)^2\}.$$

Therefore

$$M \dot{f}_1 = \frac{d}{dt} \frac{\partial T}{\partial \dot{x}^1} - \frac{\partial T}{\partial x^1} = M [\ddot{x}^1 - x^1 (\dot{x}^2)^2 - x^1 (\sin x^2 \dot{x}^3)^2],$$

$$M \dot{f}_2 = \frac{d}{dt} \frac{\partial T}{\partial \dot{x}^2} - \frac{\partial T}{\partial x^2} = M \left[\frac{d}{dt} \{ (x^1)^2 \dot{x}^2 \} - (x^1)^2 \sin x^2 \cos x^2 (\dot{x}^3)^2 \right].$$

$$M \dot{f}_3 = \frac{d}{dt} \frac{\partial T}{\partial \dot{x}^3} - \frac{\partial T}{\partial x^3} = M \left[\frac{d}{dt} \{ (x^1 \sin x^2)^2 \dot{x}^3 \} \right],$$

The equations of motion can be written down immediately if we know the components Q_r of the force system. These are obtained in any special problem as the coefficients of the linear form δW or by (8) if the force system is conservative.

2. Show that the velocity vector is given by $\frac{1}{M} g^{rs} \frac{\partial T}{\partial \dot{x}^s}$.

3. If φ is an invariant function of x^r and \dot{x}^r , show that $\frac{\partial \varphi}{\partial \dot{x}^r}$ is a covariant vector.

$$\left[\text{Show that } \frac{\partial \dot{x}^s}{\partial \dot{x}^r} = \frac{\partial x^s}{\partial x^r} \text{ and hence } \frac{\partial \varphi}{\partial \dot{x}^r} = \frac{\partial \varphi}{\partial \dot{x}^s} \frac{\partial x^s}{\partial x^r} \right]$$

4. If φ is an invariant function of x^r and \dot{x}^r , show that

$$\frac{d}{dt} \frac{\partial \varphi}{\partial \dot{x}^r} - \frac{\partial \varphi}{\partial x^r}$$

is a covariant vector.

5. Show that the conditions that a force vector Q_r be conservative are that $Q_{r,s} - Q_{s,r} = 0$.

6. *Integral of Energy.* Deduce from the equations of motion the formula

$$T + V = h,$$

where h is a constant. This relation is called *the equation of energy*, and h is *the constant of energy*.

$$\left[\frac{dT}{dt} = \frac{d}{dt} \left(\frac{M}{2} g_{mn} \dot{x}^m \dot{x}^n \right) = M g_{mn} \dot{x}^m \ddot{x}^n = M \dot{f}_m \dot{x}^m = - \frac{\partial V}{\partial x^m} \dot{x}^m = - \frac{dV}{dt} \right]$$

Hence on integration $T = h - V$.

3. Particle on a curve.

Let us suppose that the equations of a curve are

$$x^r = x^r(s), \quad \dots \dots \dots (14)$$

s being the arc of the curve. We shall suppose that the curve is smooth and that a particle is constrained to move on it. Our object is to find the equations of the constrained motion.

If we denote by λ^r, μ^r, ν^r the tangent, principal normal and binormal unit vectors and by κ, τ the curvature and torsion of the curve, these quantities are connected by the Frenet formulae (p. 159)

$$\lambda^r = \frac{dx^r}{ds}, \quad \frac{\delta \lambda^r}{\delta s} = \kappa \mu^r, \quad \frac{\delta \mu^r}{\delta s} = \tau \nu^r - \kappa \lambda^r, \quad \frac{\delta \nu^r}{\delta s} = -\tau \mu^r. \quad (15)$$

Also, if we denote the magnitude of the velocity by v , we have

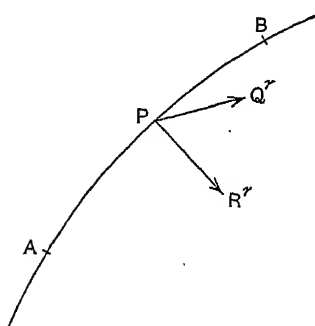
$$v^r = \frac{dx^r}{dt} = \frac{dx^r}{ds} \frac{ds}{dt} = v \lambda^r,$$

since $v = \frac{ds}{dt}$. Taking the intrinsic derivative of this equation with respect to t , we get

$$\begin{aligned} f^r &= \frac{\delta v^r}{\delta t} = \frac{dv}{dt} \lambda^r + v \frac{\delta \lambda^r}{\delta t} \\ &= \frac{dv}{dt} \lambda^r + v \frac{\delta \lambda^r}{\delta s} \frac{ds}{dt}, \end{aligned}$$

that is, using (15)

$$f^r = \frac{dv}{dt} \lambda^r + v^2 \kappa \mu^r. \quad \dots \dots \dots (16)$$



Thus the acceleration vector is coplanar with the tangent and principal normal, and its components along these directions are \dot{v} and κv^2 .

Let Q^r be the external force acting on the particle in addition to the reaction R^r of the curve. Then $(Q^r + R^r)$ is the total force vector acting on the particle and the equations of motion read

$$M f^r = Q^r + R^r,$$

that is,
$$Q^r + R^r = M \frac{dv}{dt} \lambda^r + M v^2 \kappa \mu^r.$$

Now
$$\frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt} = v \frac{dv}{ds} = \frac{1}{2} \frac{d}{ds} (v^2),$$

and the kinetic energy T is $\frac{1}{2} M v^2$. Hence

$$Q^r + R^r = \frac{dT}{ds} \lambda^r + 2T \kappa \mu^r. \quad \dots \dots \dots (17)$$

Since the curve is smooth, R^r is orthogonal to the curve, and thus $R^r \lambda_r = 0$. Multiplying (17) by λ_r, μ_r and ν_r successively, we therefore obtain the equations

$$\left. \begin{aligned} Q^r \lambda_r &= \frac{dT}{ds}, \\ Q^r \mu_r + R^r \mu_r &= 2T\kappa, \\ Q^r \nu_r &= -R^r \nu_r. \end{aligned} \right\} \dots \dots (18)$$

From the first of equations (18), we have

$$T = \int Q^r \lambda_r ds = \int Q_r \frac{dx^r}{ds} ds = - \int \frac{dV}{ds} ds$$

or

$$T + V = h, \dots \dots (19)$$

h being a constant. This last is called *the equation of energy*.

If the curve is a natural trajectory of the particle, $R^r = 0$, and we see from (17) that *the force vector must be coplanar with the tangent and principal normal of the curve*. Also

$$Q^r = \frac{dT}{ds} \lambda^r + 2T\kappa \mu^r, \dots \dots (20)$$

together with the integral of energy.

EXAMPLES.

1. If the force vector is tangent to the natural trajectory of a particle at each point prove that the path must be a straight line.

[Here $\kappa = 0$. A curve which is tangent at each point to the force vector is called *a line of force*.]

2. If a particle is made to describe a line of force show that the normal reaction is along the principal normal of the curve and equals $2(h - V)\kappa$.

3. Show that for a natural trajectory the magnitude of the force vector is

$$\left\{ \left(\frac{dT}{ds} \right)^2 + 4T^2 \kappa^2 \right\}^{\frac{1}{2}}.$$

4. If V is the potential function, show that for a natural trajectory

$$\kappa = - \frac{\partial}{\partial x^r} (\log |\overline{h - V}|) \mu^r,$$

$$2T\kappa\tau = Q_{r,s} \nu^r \lambda^s.$$

4. Particle on a surface.

We next consider the motion of a particle on a smooth surface. Let (u^1, u^2) be curvilinear coordinates on the surface and let the equations of the surface be

$$x^r = x^r(u^1, u^2). \quad (21)$$

We shall use the notation of Chapter XV (p. 193).

We have

$$v^r = \frac{dx^r}{dt} = \frac{\partial x^r}{\partial u^\alpha} \frac{du^\alpha}{dt} = x^r_\alpha \dot{u}^\alpha.$$

Therefore, writing

$$v^\alpha = \dot{u}^\alpha,$$

we have

$$v^r = x^r_\alpha v^\alpha. \quad (22)$$

We shall call v^α the *surface velocity vector* and v^r the *space velocity vector*. Differentiating (22) intrinsically with respect to t , it follows that

$$\dot{v}^r = \frac{\partial v^r}{\partial t} = x^r_\alpha \frac{\partial v^\alpha}{\partial t} + x^r_{\alpha\beta} v^\alpha v^\beta.$$

If we put $\dot{v}^\alpha = \frac{\partial v^\alpha}{\partial t}$ and use Gauss's formulae (p. 200), this becomes

$$\dot{v}^r = x^r_\alpha \dot{v}^\alpha + b_{\alpha\beta} v^\alpha v^\beta \xi^r, \quad (23)$$

where $b_{\alpha\beta}$ are the coefficients of the second groundform of the surface and ξ^r is the unit surface normal. We shall call \dot{v}^α the *surface acceleration vector*.

Let Q^r be the external force vector acting on the particle and let R^r be the reaction of the surface, which will be normal to the surface. Then the equations of motion are

$$\begin{aligned} Q^r + R^r &= M \dot{v}^r = M x^r_\alpha \dot{v}^\alpha + M b_{\alpha\beta} v^\alpha v^\beta \xi^r \\ &= M x^r_\alpha \dot{v}^\alpha + M v^2 b_{\alpha\beta} \lambda^\alpha \lambda^\beta \xi^r, \end{aligned}$$

λ^α being the surface tangent vector of the curve.

When we multiply this equation by ξ_r we get

$$Q^r \xi_r + R^r \xi_r = 2 T b_{\alpha\beta} \lambda^\alpha \lambda^\beta. \quad (24)$$

Also, multiplying the same equation by $g_{rs} x^s_\beta$ and remembering that R^r is normal to the surface, it becomes

$$g_{rs} Q^r x^s_\beta = M g_{rs} x^r_\alpha x^s_\beta \dot{v}^\alpha = M a_{\alpha\beta} \dot{v}^\alpha = M \dot{v}_\beta.$$

If we denote the surface vector $Q_r x_\alpha^r$ by Q_α (cf. § 2, p. 194 and Ex. 4, p. 196), this reads

$$Q_\beta = M f_\beta. \quad \dots \dots \dots (25)$$

Let us first examine the quantities Q_α . If the particle is given a small displacement δu^α on the surface, the amount of work done is

$$\delta W = Q_r \delta x^r = Q_r x_\alpha^r \delta u^\alpha = Q_\alpha \delta u^\alpha.$$

Hence, if the force system is conservative,

$$Q_\alpha = - \frac{\partial V}{\partial u^\alpha} \quad (26)$$

V being the potential function.

Again the kinetic energy of the particle may be written

$$T = \frac{1}{2} M a_{\alpha\beta} v^\alpha v^\beta = \frac{1}{2} M a_{\alpha\beta} \dot{u}^\alpha \dot{u}^\beta,$$

and we can easily prove (cf. p. 222) that

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{u}^\alpha} \right) - \frac{\partial T}{\partial u^\alpha} &= M a_{\alpha\beta} \left(\ddot{u}^\beta + \left\{ \begin{matrix} \beta \\ \mu \nu \end{matrix} \right\} \dot{u}^\mu \dot{u}^\nu \right) \\ &= M a_{\alpha\beta} f^\beta, \end{aligned}$$

Hence

$$M f_\alpha = \frac{d}{dt} \frac{\partial T}{\partial \dot{u}^\alpha} - \frac{\partial T}{\partial u^\alpha} \quad (27)$$

Consequently the equations of motion may be put in the form

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{u}^\alpha} - \frac{\partial T}{\partial u^\alpha} = Q_\alpha = - \frac{\partial V}{\partial u^\alpha}, \quad \dots \dots (28)$$

or equivalently

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{u}^\alpha} - \frac{\partial L}{\partial u^\alpha} = 0, \quad \dots \dots (29)$$

where L is the Lagrangian function.

The student should compare these equations with the equations (11) and (12), p. 222, for a free particle in space.

EXAMPLES.

1. If σ is the geodesic curvature and ϱ^α the normal vector of the trajectory in the surface, prove that

$$f^\alpha = \frac{1}{2} \frac{dv^2}{ds} \lambda^\alpha + \sigma v^2 \varrho^\alpha.$$

Since λ^α is the unit tangent vector to the curve,

$$v^\alpha = v \lambda^\alpha.$$

Differentiating this equation intrinsically with respect to t , we get

$$f^\alpha = \frac{\delta v^\alpha}{\delta t} = \frac{dv}{dt} \lambda^\alpha + v \frac{\delta \lambda^\alpha}{\delta t} = \frac{dv}{dt} \lambda^\alpha + v^2 \frac{\delta \lambda}{\delta s}.$$

Referring back to § 10, p. 184, we see that

$$\frac{\delta \lambda^\alpha}{\delta s} = \sigma \varrho^\alpha.$$

Also

$$\frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt} = v \frac{dv}{ds}$$

and we conclude that

$$f^\alpha = \frac{1}{2} \frac{d}{ds} (v^2) \lambda^\alpha + \sigma v^3 \varrho^\alpha.$$

2. Deduce from Ex. 1 that

$$\varrho^\alpha = \frac{dT}{ds} \lambda^\alpha + 2\sigma T \varrho^\alpha.$$

3. If there is no force system, show that $\sigma = 0$, that is, the particle describes a geodesic on the surface.

4. If the surface is described with no reaction, show that the normal component of the force vector must be $M b_{\alpha\beta} v^\alpha v^\beta$.

5. If the particle describes an asymptotic line of the surface, prove that the reaction must be equal and opposite to the normal component of the force vector. Show that the converse is also true.

5. The principle of least action. Trajectories as geodesics.

We have seen that a particle of mass M and acted upon by a force system of potential V has for its equations of motion

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{x}^r} - \frac{\partial T}{\partial x^r} = - \frac{\partial V}{\partial x^r}, \quad \dots \dots (30)$$

T being the kinetic energy, and we have also the relation

$$T + V = h, \quad \dots \dots (31)$$

where h is a constant called the constant of energy.

Let us consider all curves passing through two fixed points A and B . The integral

$$A = \sqrt{2M} \int_A^B \left\{ (h - V) g_{mn} \frac{dx^m}{d\lambda} \frac{dx^n}{d\lambda} \right\}^{\frac{1}{2}} d\lambda, \quad \dots (32)$$

taken along any one of the curves, λ being a parameter along it and h a constant, has a definite value which is called *the action* along the curve from A to B . We wish to prove the following result called *the Principle of Stationary Action*:

Of all curves passing through A and B that one for which the action is stationary is the natural trajectory of the particle which moves under the action of the force system V with h as the constant of energy.

It is a well-known result in the calculus of variations that the equations of the curve through A and B , for which the integral

$$\int_A^B \varphi(x, x') d\lambda, \quad x'^r = \frac{dx^r}{d\lambda}, \quad (33)$$

is stationary, are

$$\frac{d}{d\lambda} \frac{\partial \varphi}{\partial x'^r} - \frac{\partial \varphi}{\partial x^r} = 0. \quad (34)$$

This result may be proved by the same method as was used to obtain the equations of a geodesic on a surface (p. 171).

It follows that the curve, for which the action is stationary, is given by putting

$$\varphi = \{(h - V) g_{mn} x'^m x'^n\}^{\frac{1}{2}}$$

in (34). This gives the equations

$$\frac{d}{d\lambda} \left\{ \frac{(h - V) g_{rs} x'^s}{\varphi} \right\} - \frac{h}{2\varphi} \frac{\partial g_{mn}}{\partial x^r} x'^m x'^n + \frac{1}{2} \frac{\partial V}{\partial x^r} \frac{x'^m x'^n}{\varphi} = 0.$$

In these equations let us introduce a new variable t defined by

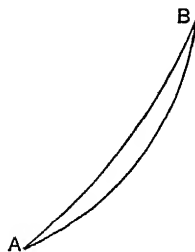
$$\frac{dt}{d\lambda} = \frac{1}{\sqrt{2}} \frac{\overline{M} \varphi}{(h - V)}, \quad \dots \dots \dots (35)$$

and we get

$$M \frac{d}{dt} (g_{rs} \dot{x}^s) - \frac{M}{2} \frac{\partial g_{mn}}{\partial x^r} \dot{x}^m \dot{x}^n + \frac{\partial V}{\partial x^r} = 0,$$

which are exactly the equations (30) where t is the time. Hence the curve giving the stationary action is the natural trajectory of a particle which moves under the action of a force potential V . Also equation (35), when squared, gives

$$M g_{mn} \dot{x}^m \dot{x}^n = 2(h - V)$$



and this shows that h is the constant of energy along the trajectory.

We shall now give a geometrical representation of the motion in which the trajectories will be represented by geodesics. Let us take a three-dimensional space the points of which are put in one-to-one correspondence with those of ordinary space, so that we may also take x^r as the coordinates of the new space. Moreover we take the line-element of this space to be given by the formula

$$dS^2 = 2M(h - V)ds^2 = 2M(h - V)g_{mn}dx^m dx^n, \quad (36)$$

ds being the line-element of ordinary space.

Now every curve in ordinary space corresponds to a curve in the new space and the action along a curve becomes

$$A = \int_A^B \frac{dS}{d\lambda} d\lambda = [S]_A^B.$$

Hence the action equals the *length* of the representative curve in the new space. Also the natural trajectory is an extremal of the variational equation

$$\delta A = \delta S = 0.$$

But the extremals of this equation are obviously the curves of stationary length or the geodesics of the representative space. *Consequently natural trajectories correspond to geodesics in our geometrical representation.* The search for the trajectories of a particle under a force potential V is therefore exactly the same problem as that of finding the geodesics of the space whose line-element is (36).

EXAMPLES.

1. Show that the action equals $2M \int_A^B \sqrt{h - V} ds$.
2. Show that the action, if calculated along a natural trajectory, equals $2 \int_A^B T dt$.

3. Show that of all curves on a surface passing through two given points A and B of the surface that one for which $\int_A^B \sqrt{h - V} ds$ is stationary is the path of the particle which moves on the surface under the action of the force potential V .

[This is the principle of stationary action for a particle on a surface. Use curvilinear coordinates on the surface together with equations (28), p. 227.]

4. Show that the paths of a particle moving on a surface of line-element $[a_{\alpha\beta} du^\alpha dv^\beta]^{\frac{1}{2}}$ under the action of a force potential V are represented by the geodesics of a surface whose line-element is $[(h - V) a_{\alpha\beta} du^\alpha dv^\beta]^{\frac{1}{2}}$.

EXAMPLES XVII.

1. If a particle is moving with uniform velocity in a straight line prove that $\frac{\delta v^r}{\delta t} = 0$.

2. If the acceleration is constant in direction but not necessarily in magnitude show that

$$\frac{\delta f^r}{\delta t} = k f^r,$$

where $k = \frac{d}{dt}(\log f)$. Show that the converse is also true.

3. *Physical components of forces.* If $\varepsilon_{(1)}^r, \varepsilon_{(2)}^r, \varepsilon_{(3)}^r$ are the unit vectors tangent to the coordinate curves, show that a force vector Q^r may be expressed in the form

$$Q^r = Q^{(1)} \varepsilon_{(1)}^r + Q^{(2)} \varepsilon_{(2)}^r + Q^{(3)} \varepsilon_{(3)}^r = Q^{(s)} \varepsilon_{(s)}^r,$$

where

$$Q^{(1)} = \sqrt{g_{11}} Q^1, \quad Q^{(2)} = \sqrt{g_{22}} Q^2, \quad Q^{(3)} = \sqrt{g_{33}} Q^3.$$

[If the force Q^r is resolved into three forces along the tangents to the coordinate curves according to the ordinary laws of the resolution of forces, then $Q^{(1)}, Q^{(2)}, Q^{(3)}$ are the three resolved forces. We may call them the *physical components* of Q^r to distinguish them from the *tensor components*, which are Q^1, Q^2, Q^3 . To prove the above result use the formulae:

$$\varepsilon_{(1)}^r = \delta_1^r / \sqrt{g_{11}}, \quad \varepsilon_{(2)}^r = \delta_2^r / \sqrt{g_{22}}, \quad \varepsilon_{(3)}^r = \delta_3^r / \sqrt{g_{33}}.]$$

4. *Parallel force system.* If the field of force forms a constant parallel vector field (as in the case of gravity), show that $Q^r_{;s} = 0$.

Hence show that a natural trajectory under such a force system satisfies the relations

$$\tau = 0, \quad \frac{d^2}{ds^2} \left(\kappa^{-\frac{2}{3}} \right) = 2\kappa^{\frac{4}{3}}, \quad T = a\kappa^{-\frac{2}{3}}.$$

Prove that these are the intrinsic equations of a *parabola* and show also that κ is a maximum at the point where Q^r is orthogonal to the curve.

5. *Bonnet's theorem:* If a curve be a natural trajectory for each of N fields of force, the velocities at any point P being $v_1, v_2, v_3, \dots, v_N$, then the curve is a natural trajectory for a field of force composed of all the separate fields of force, the velocity of P being $(v_1^2 + v_2^2 + \dots + v_N^2)^{\frac{1}{2}}$.

6. A particle moves on a surface under the action of a force, show that the sum of the normal reaction of the surface and the normal component of the force equals $Mv^2\kappa$, where κ is the normal curvature of the surface in the direction of motion.

7. *The momentum vector.* If we write $p_r = \frac{\partial T}{\partial \dot{x}^r}$, show that p_r is a covariant vector and prove that

$$T = \frac{1}{2M} g^{mn} p_m p_n.$$

8. *Hamilton's equations of motion.* If we put $H = T + V$ and express H in terms of x^r and p_r , show that the equations of motion can be put in the form

$$\frac{dx^r}{dt} = \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} = -\frac{\partial H}{\partial x^r}.$$

9. If H is an invariant function of x^r and p_r only, show that $\frac{\partial H}{\partial p_r}$ is a contravariant vector and that $\left(\frac{dp_r}{dt} + \frac{\partial H}{\partial x^r}\right)$ is a covariant vector. Deduce that *Hamilton's equations are vector equations.*

[For a new coordinate system \bar{x}^r , we have $p_r = \bar{p}_s \frac{\partial \bar{x}^s}{\partial x^r}$, that is, $\frac{\partial p_r}{\partial \bar{p}_s} = \frac{\partial \bar{x}^s}{\partial x^r}$. Hence $\frac{\partial \bar{H}}{\partial \bar{p}_r} = \frac{\partial H}{\partial p_s} \frac{\partial p_s}{\partial \bar{p}_r} = \frac{\partial H}{\partial p_s} \frac{\partial \bar{x}^s}{\partial x^r}$, which proves the first result. The second is proved in a similar manner.]

10. Prove that $\frac{\delta p_r}{\delta t} = -\frac{\partial V}{\partial x^r}$.

11. If we put $L = T - V$ and express L in terms of x^r, \dot{x}^r whilst H is expressed in terms of x^r, p_r , prove that

$$\frac{\partial H}{\partial x^r} = -\frac{\partial L}{\partial x^r}.$$

12. Particles, of equal masses M and acted upon by a certain force system, start off with equal velocities v in all directions orthogonal to a given direction. Show that the surface on which the different trajectories lie has an umbilic at the initial point.

13. For the motion of a particle on a surface, show that if we put

$$p_\alpha = M a_{\alpha\beta} \dot{u}^\beta, \quad H = \frac{1}{2M} a^{\alpha\beta} p_\alpha p_\beta + V$$

the equations of motion on the surface may be written

$$\frac{du^\alpha}{dt} \frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{dt} + \frac{\partial H}{\partial u^\alpha} = 0.$$

[These are Hamilton's equations for the motion of a particle on a surface.]

14. If S is an invariant function of $(x^r, \dot{x}^r, \ddot{x}^r)$ show that $\frac{\partial S}{\partial \dot{x}^r}$ is a covariant vector.

15. *Appell's equations.* If we put $S = \frac{1}{2} M g_{mn} \dot{f}^m \dot{f}^n$, show that the equations of motion can be put in the form

$$\frac{\partial S}{\partial \dot{x}^r} = Q_r = -\frac{\partial V}{\partial x^r}.$$

CHAPTER XVIII

DYNAMICS OF RIGID BODIES

Section A.

RECTILINEAR COORDINATES

1. Moments of Inertia.

Let us suppose that we are given N particles, of which a typical one has mass $M_{(\alpha)}$ and coordinates $x_{(\alpha)}^r$ referred to a set of rectilinear coordinate axes. If we regard the particles as being rigidly connected the rigid body will be included as a particular case.

The coordinates of the centre of gravity of the system of particles is given by

$$\bar{x}^r = \frac{\sum_{\alpha=1}^N M_{(\alpha)} x_{(\alpha)}^r}{\sum_{\alpha=1}^N M_{(\alpha)}}. \quad \dots \quad (1)$$

If we are considering a continuous distribution of matter the summations in this formula and in subsequent ones ought to be replaced by triple integrals, the element of mass dM taking the place of $M_{(\alpha)}$.

Another quantity which is of importance in the dynamics of a system of particles is the *double symmetric tensor*

$$I^{rs} \equiv \sum_{\alpha=1}^N M_{(\alpha)} x_{(\alpha)}^r x_{(\alpha)}^s, \quad \dots \quad (2)$$

which we call the *inertia tensor relative to the origin*.

If we are given a straight line through O and perpendiculars $p_{(\alpha)}$ are drawn from the different particles on the line, the quantity $\sum_{\alpha=1}^N M_{(\alpha)} p_{(\alpha)}^2$ is called the *moment of inertia of the system of particles round the given line*. We can express this moment of inertia in terms of I^{rs} . For, if λ^r is the unit vector in the direction of the line,

$$p_{(\alpha)}^2 = g_{mn} x_{(\alpha)}^m x_{(\alpha)}^n - (g_{mn} \lambda^m x_{(\alpha)}^n)^2.$$

Hence the moment of inertia round λ^r is

$$I(\lambda^r) = \sum_{\alpha=1}^N M_{(\alpha)} p_{(\alpha)}^2 = g_{mn} I^{mn} - g_{mp} g_{nq} \lambda^m \lambda^n I^{pq}.$$

If we denote by I_{rs} the associated tensor $g_{rm} g_{sn} I^{mn}$ and by I the invariant $g_{mn} I^{mn}$, this becomes

$$I(\lambda^r) = (I g_{rs} - I_{rs}) \lambda^r \lambda^s, \quad (3)$$

since λ^r is a unit vector. Therefore if we consider the quadric Q whose equation is

$$(I g_{rs} - I_{rs}) x^r x^s = 1, \quad \dots \dots (4)$$

we find that the moment of inertia round λ^r is $1/R^2$, where R is the radius vector of Q in the direction of λ^r . Q is called the *momental ellipsoid* at O .

We see from § 3, p. 107, that the momental ellipsoid at O has always three principal axes, which are called the *principal axes of inertia* at O , and the planes containing them in pairs are called the *principal planes of inertia* at O . If these axes be taken as coordinate axes then the axes are orthogonal and at the same time

$$I_{12} = I_{23} = I_{31} = 0.$$

These principal axes are given by the equations

$$(I g_{rs} - I_{rs}) \lambda^s = \theta \lambda_r, \quad \dots \dots (5)$$

where θ is a root of the determinantal equation

$$(I - \theta) g_{rs} - I_{rs} = 0. \quad \dots \dots (6)$$

EXAMPLES.

1. Show that the moment of inertia round any straight line equals the moment of inertia round a parallel straight line through the centre of gravity plus M times the square of the distance between the lines, where M is the total mass of the particles.

[The moment of inertia round a line through the centre of gravity ξ^r and in the direction λ^r is $\{(I - M g_{mn} \xi^m \xi^n) g_{rs} - I_{rs} + M \xi_r \xi_s\} \lambda^r \lambda^s$.]

2. If $p_{(a)}$ is the perpendicular from $M_{(a)}$ on a given plane, $\Sigma M_{(a)} p_{(a)}^2$ is called the *moment of inertia with respect to the plane*. Show that the moment of inertia with respect to the plane $u_r x^r = 0$ is $I^{mn} u_m u_n / g^{mn} u_m u_n$.

3. Prove that the moment of inertia with respect to the plane $u_r x^r = 1$ is $(I^{mn} u_m u_n - 2 M u_m \xi^m + M) / g^{mn} u_m u_n$.

4. Show that I^{33}/g^{33} is the moment of inertia with respect to the x^3 -plane. Similarly interpret I^{11} and I^{22} .

5. If I_3 is the moment of inertia round the x^3 -axis, show that $I_3 = I - I_{33}/g_{33}$. If, further, the axes are Cartesian and orthogonal show that $I_3 = I_{11} + I_{22}$.

2. The equations of motion.

We now wish to obtain the equations of motion of a system of particles and more especially of a rigid body under the action of given forces.

If $X'_{(a)}$ is the total force vector acting on the particle $M_{(a)}$ we know that

$$M_{(a)}\ddot{x}''_{(a)} = X'_{(a)},$$

from the equations of motion of a single particle. Hence if $\delta x''_{(a)}$ is any small displacement given to this particle we have

$$\{M_{(a)}\ddot{x}''_{(a)} - X'_{(a)}\} \delta x_{(a)r} = 0, \quad (7)$$

where $\delta x_{(a)r}$ is the associated vector $g_{rs} \delta x''_{(a)}$. The expression $X'_{(a)} \delta x_{(a)r}$, which equals the product of the force and the projection of the displacement in the direction of the force, is *the work done by $X'_{(a)}$ in the displacement $\delta x''_{(a)}$* .

Let us sum the expressions (7) for all the particles. We have

$$\sum_{\alpha=1}^N \{M_{(a)}\ddot{x}''_{(a)} - X'_{(a)}\} \delta x_{(a)r} = 0. \quad (8)$$

In this expression we have not so far restricted the displacements $\delta x''_{(a)}$, and the forces $X'_{(a)}$ are all the forces, external and internal, that act on the individual particles. If we restrict ourselves to certain types of displacements there may be certain of the forces which do no work under these displacements and consequently the part of the sum $\sum_{\alpha=1}^N X'_{(a)} \delta x_{(a)r}$ due to these forces must be zero. For example, if we consider the particles as forming a rigid body and if we give to the system only rigid body displacements the totality of the internal forces do no work and can therefore be neglected in the equation (8).

Consequently for a rigid body we have

$$\sum_{\alpha=1}^N \{M_{(a)}\ddot{x}''_{(a)} - X'_{(a)}\} \delta x_{(a)r} = 0, \quad (9)$$

where $\delta x''_{(a)}$ refer only to rigid body displacements and $X'_{(a)}$ are the external forces alone. Now we see from § 5, p. 126, that the most general infinitesimal rigid body displacement is of the form

$$\delta x'' = u^r \delta t + \omega^r_{,s} x^s \delta t, \quad (10)$$

where ω_{rs} is skew-symmetric. Substituting in (9), we get

$$\sum_{\alpha} \{M_{(\alpha)} \ddot{x}_{(\alpha)}^r - X_{(\alpha)}^r\} u_r + \sum_{\alpha} \{M_{(\alpha)} \ddot{x}_{(\alpha)}^r - X_{(\alpha)}^r\} \omega_{rs} x_{(\alpha)}^s = 0.$$

We can always find a vector ω^r such that $\omega_{rs} = -\omega^p \varepsilon_{p rs}$; in fact, $\omega^r = -\varepsilon^{mnr} \omega_{mn}/2$. Hence the equation becomes

$$\sum_{\alpha} \{M_{(\alpha)} \ddot{x}_{(\alpha)}^r - X_{(\alpha)}^r\} u_r + \sum_{\alpha} \varepsilon_{rst} \{M_{(\alpha)} \ddot{x}_{(\alpha)}^t - X_{(\alpha)}^t\} x_{(\alpha)}^s \omega^r = 0.$$

This equation is true for arbitrary values of u_r and ω^r and it is therefore equivalent to the equations

$$\begin{aligned} \sum_{\alpha} M_{(\alpha)} \ddot{x}_{(\alpha)}^r &= \sum_{\alpha} X_{(\alpha)}^r, \\ \sum_{\alpha} M_{(\alpha)} \varepsilon_{rst} x_{(\alpha)}^s \ddot{x}_{(\alpha)}^t &= \sum_{\alpha} \varepsilon_{rst} x_{(\alpha)}^s X_{(\alpha)}^t, \end{aligned}$$

or, as they may be written,

$$\left. \begin{aligned} \frac{d}{dt} \left\{ \sum_{\alpha} M_{(\alpha)} \dot{x}_{(\alpha)}^r \right\} &= \sum_{\alpha} X_{(\alpha)}^r = X^r, \\ \frac{d}{dt} \left\{ \sum_{\alpha} M_{(\alpha)} \varepsilon_{rst} x_{(\alpha)}^s \dot{x}_{(\alpha)}^t \right\} &= \sum_{\alpha} \varepsilon_{rst} x_{(\alpha)}^s X_{(\alpha)}^t. \end{aligned} \right\} \quad \dots \quad (11)$$

The vector

$$G^r \equiv \sum_{\alpha} M_{(\alpha)} \dot{x}_{(\alpha)}^r \quad \dots \quad (12)$$

is called *the linear momentum vector* and, if M is the total mass and ξ^r are the coordinates of the centre of gravity, we see that

$$G^r = M \dot{\xi}^r.$$

Also

$$H_r \equiv \sum_{\alpha} M_{(\alpha)} \varepsilon_{rst} x_{(\alpha)}^s \dot{x}_{(\alpha)}^t \quad \dots \quad (13)$$

is called *the angular momentum vector relative to the origin*, whilst the vector

$$L_r \equiv \sum_{\alpha} \varepsilon_{rst} x_{(\alpha)}^s X_{(\alpha)}^t \quad \dots \quad (14)$$

is called *the moment vector of the system of forces $X_{(\alpha)}^r$ relative to the origin*. The equations (11) now read

$$\left. \begin{aligned} \frac{d}{dt} G^r &= X^r, \\ \frac{d}{dt} H_r &= L_r. \end{aligned} \right\} \quad \dots \quad (15)$$

We shall next find an expression for H_r in terms of the inertia tensor of the system. We see from (10) that the velocity of any point of a rigid body can be put in the form

$$\dot{x}^r = u^r + \omega^r_s x^s,$$

u^r representing the linear velocity of the point coinciding for the moment with the origin and ω^r_s representing the angular velocity. Hence

$$\begin{aligned} H_r &= \sum_a M_{(a)} \varepsilon_{rst} x_{(a)}^s \dot{x}_{(a)}^t = \sum_a M_{(a)} \varepsilon_{rst} x_{(a)}^s (u^t + \omega^t_p x_{(a)}^p) \\ &= M \varepsilon_{rst} \xi^s u^t + \varepsilon_{rst} \omega^t_p I^s p. \end{aligned}$$

If we put $\omega^{rs} = -\varepsilon^{nrs} \omega_n$, we have

$$H_r = M \varepsilon_{rst} \xi^s u^t - (I_r^s - I_s^r) \omega_s. \quad \dots \dots (16)$$

We shall consider the two following particular cases:

(i) If the rigid body has one point fixed and if we take this point to be the origin, then $u^r = 0$ and

$$H_r = (I \delta_r^s - I_s^r) \omega_s.$$

(ii) If we take the origin as coinciding with the centre of gravity for the moment, then $\xi^r = 0$ and

$$H_r = (I \delta_r^s - I_s^r) \omega_s.$$

EXAMPLES.

1. Show that the motion of the centre of gravity is given by the equations

$$M \ddot{\xi}^r = X^r,$$

that is, *the centre of gravity moves as if the whole mass were concentrated at this point and all the forces were acting at it parallel to their actual directions.*

2. The angular momentum vector relative to a point x_0^r is defined as

$$H_r^0 \equiv \sum_a M_{(a)} \varepsilon_{rst} (x_{(a)}^s - x_0^s) \dot{x}_{(a)}^t.$$

Show that

$$H_r^0 = \sum_a M_{(a)} \varepsilon_{rst} x_{(a)}^s \dot{x}_{(a)}^t - M \varepsilon_{rst} x_0^s \dot{\xi}^t.$$

3. The moment vector of the forces $X_{(a)}^r$ relative to x_0^r is defined as

$$L_r^0 = \sum_a \varepsilon_{rst} (x_{(a)}^s - x_0^s) X_{(a)}^t.$$

Show that

$$L_r^0 = \sum_a \varepsilon_{rst} x_{(a)}^s X_{(a)}^t - \varepsilon_{rst} x_0^s X^t.$$

4. Prove that

$$\frac{dH_r^0}{dt} = -M \varepsilon_{rst} \dot{x}_0^s \dot{\xi}^t.$$

5. Deduce from Ex. 4 that we have the equations

$$\frac{dH_r^0}{dt} = L_r^0$$

if the point x_0^r is the centre of gravity. *Consequently the motion relative to the centre of gravity is the same as if the centre of gravity were fixed and the body subjected to the action of the same forces.*

3. Moving axes. Euler's equations.

We shall divide the discussion of the motion of a rigid body into two cases.

(i) There is a point of the body fixed in space. We shall take this point as the origin and we find that equations (15) hold. In the first of these we must include in X^r the reaction at the fixed point. Also we can replace H_r by $(I\delta_r^s - I_r^s)\omega_s$, where I_r^s is the mixed inertia tensor, I the invariant I_m^m and ω^r is the angular velocity vector of the body.

(ii) There is no point of the body fixed in space. The equations (15) still hold referred to axes fixed in space. We see that the second set of (15) also hold if H_r is the moment of momentum vector and L_r is the moment vector of the force system relative to the centre of gravity. In other words, the motion of the body relative to the centre of gravity is exactly the same as if the centre of gravity were fixed and the same forces were acting. Hence the absolute motion of the centre of gravity is given by the first of (15) referred to axes fixed in space and the motion relative to the centre of gravity is found as in (i), the fixed point being the centre of gravity.

Now if we put

$$H_r = (I\delta_r^s - I_r^s)\omega_s \quad \dots \dots \dots (17)$$

and try to find $\frac{dH_r}{dt}$, there will be involved derivatives of I_r^s with respect to t since these are variable with the time. To avoid this we shall take moving axes fixed in the body, with respect to which the inertia tensor will have constant components.

Let \bar{x}^r be a system of coordinates referred to axes which are fixed in space and let x^r be the coordinate system which moves rigidly with the body and coincides with \bar{x}^r initially. By reference to p. 124, we see that the transformation from one set of axes to another set, which have been derived from the first by a continuous rigid-body displacement, is given by

$$\bar{x}^r = a^r_{.s} x^s, \quad \dots \dots \dots (18)$$

where $a^r_{.s}$ is a function of t satisfying the equations

$$g_{mn} \omega^n_r \alpha^m_{.s} = g_{rs}.$$

Also

$$\frac{da^r_{.s}}{dt} = a^{rp} \omega_{ps}, \quad \dots \dots \dots (19)$$

ω_{ps} being a skew-symmetric tensor and a^{rp} being $g^{ps} a^r_{.s}$.

Now a covariant vector whose components referred to the moving axes are A_r has for its components referred to the fixed axes \bar{A}_r where

$$A_r = \bar{A}_s a^s_{.r}, \quad \dots \dots \dots (20)$$

Differentiating this with respect to t , we have from (19)

$$\begin{aligned}\frac{dA_r}{dt} &= \frac{d\bar{A}_s}{dt} a^s_r + \bar{A}_s a^s_t \omega^t_r \\ &= a^s_r \frac{d\bar{A}_s}{dt} + A_t \omega^t_r,\end{aligned}$$

that is,

$$\frac{dA_r}{dt} - A_s \omega^s_r = a^s_r \frac{d\bar{A}_s}{dt} \dots \dots \dots (21)$$

We have

$$\frac{d\bar{H}_r}{dt} = \bar{L}_r$$

referred to the fixed axes. Consequently, making use of (20) and (21), we obtain

$$\frac{dH_r}{dt} - H_s \omega^s_r = L_r \dots \dots \dots (22)$$

These are the equations of motion referred to the moving axes.

The angular velocity vector ω^r of the moving axes (and of the body) is given (p. 237) by

$$\omega_{rs} = -\varepsilon_{prs} \omega^p, \dots \dots \dots (23)$$

and

$$H_r = (I \delta_r^p - I_r^p) \omega_p,$$

where I_r^r are now fixed constants. Thus

$$\frac{dH_r}{dt} = (I \delta_r^p - I_r^p) \dot{\omega}_p$$

and the equations (22) become

$$(I \delta_r^p - I_r^p) \dot{\omega}_p - \varepsilon_{rs p} (I \delta_t^s - I_t^s) \omega^t \omega^p = L_r \dots \dots (24)$$

These are *Euler's equations* in tensor form.

EXAMPLES:

1. If the moving axes are the principal axes of inertia at the fixed point show that Euler's equations become

$$I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 = L_1, \text{ etc.},$$

where $I_1 = I_2^2 + I_3^2$, etc.

[The axes are now Cartesian and rectangular and the only components of I_r^r surviving are I_1^1, I_2^2, I_3^3 . The quantities I_1, I_2, I_3 are the principal moments of inertia.]

2. If A^r is a *contravariant* vector referred to the moving axes and \bar{A}^r its components referred to the fixed axes, show that

$$\frac{d\bar{A}^r}{dt} = a^r_s \left(\frac{dA^s}{dt} + \omega^s_t A^t \right).$$

3. If A^r is a vector fixed in space show that its components referred to moving axes satisfy the equations

$$\frac{dA^r}{dt} = -\omega^r_s A^s.$$

4. If K_{rs} are the components of a double covariant tensor referred to moving axes and \bar{K}_{rs} its components referred to fixed axes, show that

$$\frac{dK_{mn}}{dt} - \omega^p_m K_{pn} - \omega^p_n K_{mp} = a^r_m a^s_n \frac{d\bar{K}_{rs}}{dt}.$$

[This result may be proved by differentiating the equation $K_{mn} = \bar{K}_{rs} a^r_m a^s_n$ and substituting from (19), or as follows: Let A^m, B^n be two vectors fixed in space. Then $K_{mn} A^m B^n$ being an invariant equals $\bar{K}_{mn} \bar{A}^m \bar{B}^n$ and their derivatives are equal. The result then follows from Ex. 3.]

Section B.

THE GEOMETRY OF DYNAMICS.

4. Generalised coordinates of a dynamical system.

A material system is regarded from the dynamical standpoint as a collection of particles which are subject to interconnections and constraints of various kinds. For example, a rigid body is regarded as a number of particles rigidly connected together so as to remain at invariable distances from each other.

When we know the constitution of such a dynamical system and the constraints to which it is subject, the configuration or position of the system will be determined by a certain number of independent variables, which are called *the coordinates of the dynamical system*. Thus a free particle in space is determined by *three* coordinates referred to a system of axes fixed in space and a particle constrained to move on a surface is determined by *two* independent coordinates, the curvilinear coordinates on the surface. Again a rigid straight bar is determined by *five* coordinates, namely three giving the position of one end of the bar in space and two others giving the direction ratios of the rod.

The number of independent coordinates which determine the configuration of a dynamical system completely is called *the number of degrees of freedom of the system*. Let us suppose that we have a dynamical system which we shall assume for simplicity to have *three* degrees of freedom. The student will

see that the ideas we are about to develop can be immediately extended to a dynamical system of any number of degrees of freedom. The configuration of our system at any time is then given by three coordinates which we denote by

$$q^1, q^2, q^3,$$

and if we keep our conventions regarding italic indices these may be written more briefly q^r . As the coordinates change in value the dynamical system of course changes its configuration.

There is obviously an infinite number of sets of independent coordinates which will determine the configuration of a dynamical system, but since the position of the system is completely given by any one set these sets of coordinates must be functionally related. Hence if \bar{q}^r is any other set of coordinates these quantities must be connected with q^r by formulae of the type

$$\bar{q}^r = f^r(q^1, q^2, q^3) \dots \dots \dots (25)$$

These equations are the equations of transformation from one set of dynamical coordinates to another and we can define in the usual way vectors and tensors relative to the coordinate transformations (25). Thus A^r_{st} are the components of a mixed triple tensor in the coordinate system q^r if its components in the new coordinate system are given by

$$\bar{A}^r_{st} = A^m_{np} \frac{\partial \bar{q}^r}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^s} \frac{\partial q^p}{\partial \bar{q}^t}, \dots \dots \dots (26)$$

the summations of the repeated indices on the right being as usual from 1 to 3.

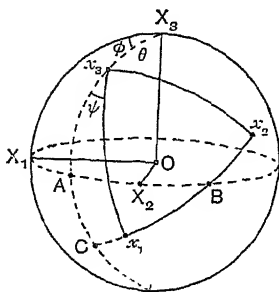
EXAMPLES.

1. *Show that a rigid body with one point fixed has three degrees of freedom.*

Let $Ox_1x_2x_3$ be the principal trihedral of inertia of the body at any instant and let $OX_1X_2X_3$ be the initial position of this trihedral, so that the former is moving with the body and the latter is fixed in space. Let us take the intersection of these trihedrals with the unit sphere of centre O , which is the fixed point of the body.

We denote by θ the angle X_3Ox_3 , by φ the angle between the planes X_1OX_3 , X_3Ox_3 and by ψ the angle between the planes X_3Ox_3 and x_3Ox_1 . These angles are called *Euler's angles* and we shall show that the position of the body is determined completely by these three angles, which will therefore act as coordinates.

Let OA be the intersection of the planes X_3Ox_3 , X_1OX_2 , OB the intersection of x_1Ox_2 , X_1OX_2 , and OC the intersection of x_1Ox_2 and X_3Ox_3 . A rotation of amount φ round OX_3 will make OX_1 coincide with OA



and OX_2 with OB . A rotation of amount θ round OB will then bring OX_3 to Ox_3 and OA to OC . Finally a rotation of amount ψ round Ox_3 will bring OC to Ox_1 and OB to Ox_2 . Consequently these three rotations performed successively will bring the trihedral $OX_1X_2X_3$ to the position $Ox_1x_2x_3$, that is, the principal axes of inertia and therefore also the position of the body are determined completely by the three quantities θ , φ , ψ .

2. Show that the transformation from one set of axes to the other is in accordance with the scheme:

	X^1	X^2	X^3
x^1	$\cos \theta \cos \varphi \cos \psi - \sin \varphi \sin \psi$	$\cos \theta \sin \varphi \cos \psi + \cos \varphi \sin \psi$	$-\sin \theta \cos \psi$
x^2	$-\cos \theta \cos \varphi \sin \psi - \sin \varphi \cos \psi$	$-\cos \theta \sin \varphi \sin \psi + \cos \varphi \cos \psi$	$\sin \theta \sin \psi$
x^3	$\sin \theta \cos \varphi$	$\sin \theta \sin \varphi$	$\cos \theta$

3. Show that a free rigid body can be determined by six coordinates, namely, the three coordinates in space of its centre of gravity and the three Eulerian angles relative to the centre of gravity.

5. The equations of motion in generalised coordinates.

If we suppose that the points of our dynamical system are referred to rectilinear axes x^r in ordinary space, then when we are given the time and a set of coordinates q^r we are also given all the points x^r of the dynamical system, for the system is determined uniquely.

Consequently the x^r are functions of q^r and possibly also of the time, that is,

$$x^r = x^r(q^1, q^2, q^3, t).$$

We shall restrict ourselves to the consideration of dynamical systems in which these equations do not involve t , so that

$$x^r = x^r(q^1, q^2, q^3). \quad \dots \dots \dots (27)$$

Differentiating (27) with respect to t , we have

$$\dot{x}^r = \frac{\partial x^r}{\partial q^s} \dot{q}^s \dots \dots \dots (28)$$

The quantities \dot{q}^r , which form a vector with reference to co-ordinate transformations (25), we shall call *the generalised velocity vector*. We see from (28) that when the generalised velocity vector is given we know the velocity of each point of our system.

The kinetic energy of the system is T where

$$T = \frac{1}{2} \sum M g_m \dot{x}^m \dot{x}^n = \frac{1}{2} \sum M g_{mn} \frac{\partial x^r}{\partial q^r} \frac{\partial x^n}{\partial q^s} \dot{q}^r \dot{q}^s.$$

Thus we may write

$$T = \frac{1}{2} a_{rs} \dot{q}^r \dot{q}^s, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where we have put

$$= \sum M g_n \frac{\partial x^m}{\partial q^r} \frac{\partial x^n}{\partial q^s}, \quad . \quad . \quad . \quad . \quad . \quad (30)$$

that is, the kinetic energy is a homogeneous quadratic form in \dot{q}^r . From (30) we see that a_{rs} is symmetric in r and s . Also, since T is an invariant for all transformations of generalised coordinates, we conclude from (29) that a_{rs} is a double symmetric tensor.

Let us next find the equations of motion of our dynamical system in terms of the generalised coordinates. Referring to equation (8), p. 235, we see that the equations of motion in rectilinear coordinates are embodied in the single equation

$$\sum g_{mn} (M \ddot{x}^m - X^m) \delta x^n = 0, \quad . \quad . \quad . \quad (31)$$

where δx^r is a virtual displacement of the system, and we can neglect in X^r all the internal or external forces which do no work in this displacement. If we give the system a small displacement compatible with the constraints of the system, we see that this displacement may be effected by giving increments δq^r to the coordinates of the system, and these are related to the δx^r in accordance with the formulae

$$\delta x^r = \frac{\partial x^r}{\partial q^s} \delta q^s$$

Moreover in this displacement the internal forces due to the constraints of the system will do no work, since these constraints are preserved, and consequently only the external forces will appear in (31).

Thus (31) becomes

$$\sum g_{mn} \left\{ M \frac{d}{dt} \left(\frac{\partial x^m}{\partial q^s} \dot{q}^s \right) \frac{\partial x^n}{\partial q^r} - X^m \frac{\partial x^n}{\partial q^r} \right\} \delta q^r = 0. \quad (32)$$

Now

$$\begin{aligned} & \sum M g_{mn} \frac{d}{dt} \left(\frac{\partial x^m}{\partial q^s} \dot{q}^s \right) \frac{\partial x^n}{\partial q^r} \\ &= \frac{d}{dt} \left\{ \sum M g_{mn} \frac{\partial x^m}{\partial q^s} \frac{\partial x^n}{\partial q^r} \dot{q}^s \right\} - \sum M g_{mn} \frac{\partial x^m}{\partial q^s} \frac{\partial^2 x^n}{\partial q^r \partial q^t} \dot{q}^s \dot{q}^t \\ &= \frac{d}{dt} (a_{rs} \dot{q}^s) - \frac{1}{2} \frac{\partial a_{st}}{\partial q^r} \dot{q}^s \dot{q}^t, \\ & \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^r} - \frac{\partial T}{\partial q^r}. \end{aligned}$$

Also, if we put

$$Q_r = \sum g_{mn} X^m \frac{\partial x^n}{\partial q^r},$$

we have

$$Q_r \delta q^r = \sum g_{mn} X^m \delta x^n = \delta W, \dots \dots (33)$$

where δW is the work done by the external forces in the small displacement δq^r , which shows that Q_r is a *covariant vector*. We shall call it the *generalised force vector*.

The equation (32) now takes the form

$$\left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^r} - \frac{\partial T}{\partial q^r} - Q_r \right\} \delta q^r = 0.$$

Since the coordinates q^r are independent this equation is true for all variations of δq^r and the equations of motion become

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^r} - \frac{\partial T}{\partial q^r} - Q_r = 0 \dots \dots \dots (34)$$

EXAMPLES.

1. Show that

$$\left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^r} - \frac{\partial T}{\partial q^r} \right\} \delta q^r = \sum M g_{mn} \ddot{x}^m \delta x^n,$$

and deduce that the quantities $\left\{ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^r} - \frac{\partial T}{\partial q^r} \right\}$ form a *covariant vector*.

2. If $[rs, t]$ is the Christoffel symbol of the first kind relative to a_{rs} , that is, if

$$[rs, t] = \frac{1}{2} \left(\frac{\partial a_{st}}{\partial q^r} + \frac{\partial a_{rt}}{\partial q^s} - \frac{\partial a_{rs}}{\partial q^t} \right),$$

show that

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^r} - \frac{\partial T}{\partial q^r} = a_{rs} \ddot{q}^s + [st, r] \dot{q}^s \dot{q}^t.$$

3. If a^{rs} is the cofactor of a_{rs} in $|a_{rs}|$, divided by this determinant, show that the equations of motion may be written

$$\ddot{q}^r + \left\{ \begin{matrix} r \\ st \end{matrix} \right\} \dot{q}^s \dot{q}^t =$$

where

$$\left\{ \begin{matrix} r \\ st \end{matrix} \right\} = a^{rp} [st, p], \quad Q^r = a^{rs} Q_s.$$

$\left\{ \begin{matrix} r \\ st \end{matrix} \right\}$ is the Christoffel symbol of the second kind relative to a_{rs} , and we see that $\ddot{q}^r + \left\{ \begin{matrix} r \\ st \end{matrix} \right\} \dot{q}^s \dot{q}^t$ is the intrinsic derivative of \dot{q}^r with respect to t . Using the notation of p. 145, it may be written $\frac{\delta \dot{q}^r}{\delta t}$ and we know that it is a contravariant vector. ⁷

4. If the force system is conservative and V is the potential function show that

$$Q_r = - \frac{\partial V}{\partial q^r}$$

and that the equations of motion take Lagrange's form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^r} - \frac{\partial L}{\partial q^r} = 0,$$

where $L = T - V$.

[The first result follows from (33) since $\delta W = -\delta V$ and the second result follows from the fact that V does not contain \dot{q}^r .]

6. The manifold of configurations.

Let us consider a space or manifold of three dimensions in which a point is determined by the three coordinates q^r . Then to each point of the manifold there corresponds a definite configuration of the dynamical system whose coordinates are q^r and we have therefore a geometrical representation of the configurations of our dynamical system. For this reason the manifold is called *the manifold of configurations*.

If the dynamical system moves in any way, its coordinates are given as functions of the time. Thus the motion is given by equations of the form

$$q^r = q^r(t). \quad (35)$$

As t varies we therefore observe that the representative point in the manifold of configurations describes a curve and (35) are the equations of this curve. Such a curve is called a *trajectory*.

If, moreover, the system is moving freely under the action of a given force system in accordance with the laws of dynamics, the corresponding point describes a curve in the manifold of configurations and this curve is called a *natural trajectory of the given force system*.

7. The kinematical line-element.

The kinetic energy of the system is

$$T = \frac{1}{2} a_{rs} \dot{q}^r \dot{q}^s, \dots \dots \dots (36)$$

where a_{rs} are functions of the coordinates q^r . Now T is always positive except when \dot{q}^r is zero in which case T vanishes. In other words, the quadratic form (36) is positive definite. Consequently we can always find ds such that

$$ds^2 = a_{rs} dq^r dq^s \dots \dots \dots (37)$$

This ds we shall *define* to be the *distance* between the neighbouring points q^r and $q^r + dq^r$ in the manifold of configurations, that is, the line-element of our manifold is defined by (37). Thus the ds^2 is a homogeneous quadratic function of dq^r just as in the case of Euclidean three-dimensional space. A space in which the ds^2 is given by a formula such as (37) is called a *Riemannian space*. We shall call the line-element (37) *the kinematical line-element* to distinguish it from another which we shall mention later.

Having thus obtained the fundamental tensor a_{rs} of our space, we define in the usual way the associated tensor a^{rs} as the cofactor of a_{rs} in $|a_{rs}|$, divided by the determinant. We define the magnitude of a vector A^r by A where

$$A^2 = a_{rs} A^r A^s,$$

and we change vectors and tensors into associated vectors and tensors as we have already done in the case of Euclidean space. We can also define intrinsic and covariant differentiation of tensors by means of the Christoffel symbols as at p. 146.

Let us suppose that a point describes a curve of the manifold and let us take the arc s of the curve as parameter, so that the equations of the curve will be

$$q^r = q^r(s) \dots \dots \dots (38)$$

We know that $\frac{dq^r}{ds}$ is a vector and we shall write it

$$\lambda^r = \frac{dq^r}{ds} \dots \dots \dots (39)$$

From (37) we conclude that

$$a_{rs} \lambda^r \lambda^s = 1, \dots \dots \dots (40)$$

that is, λ^r is a *unit* vector. We call it *the unit tangent vector of the curve*.

If we take the intrinsic derivative of equation (40) we get

$$a_{rs} \lambda^r \frac{\delta \lambda^s}{\delta s} = 0,$$

which shows that the vector $\frac{\delta \lambda^r}{\delta s}$ is orthogonal to the curve. If we denote its magnitude by κ , we may write

$$\frac{\delta \lambda^r}{\delta s} = \kappa \mu^r, \dots \dots \dots (41)$$

where μ^r is a unit vector. The quantity κ is called *the curvature of the curve* and μ^r *the principal normal of the curve* from the analogy with Euclidean space (p. 157).

8. The dynamical trajectories of the manifold of configurations.

We have seen that the equations of a trajectory are of the form (35), that is, t is usually taken as the parameter of the curve.

We shall call the vector

$$v^r = \dot{q}^r \dots \dots \dots (42)$$

the velocity vector and we see that the square of its magnitude is

$$v^2 = a_{rs} v^r v^s = 2T \dots \dots \dots (43)$$

We define *the acceleration vector* f^r as the intrinsic derivative of v^r with respect to t . The expression for f^r is therefore

$$f^r = \frac{\delta v^r}{\delta t} = \dot{v}^r + \left\{ \begin{matrix} r \\ st \end{matrix} \right\} v^s \dot{q}^t, \quad \dots \quad (44)$$

or,

$$f^r = \ddot{q}^r + \left\{ \begin{matrix} r \\ st \end{matrix} \right\} \dot{q}^s \dot{q}^t.$$

Let us now change the parameter of the trajectory from t to s , the arc of the curve. We have

$$ds^2 = a_{rs} dq^r dq^s = a_{rs} \dot{q}^r \dot{q}^s dt^2.$$

Hence

$$\frac{ds}{dt} = (a_{rs} \dot{q}^r \dot{q}^s)^{\frac{1}{2}} = v. \quad \dots \quad (45)$$

Also

$$v^r = \frac{dq^r}{dt} = \frac{dq^r}{ds} \frac{ds}{dt} = v \lambda^r. \quad (46)$$

and consequently, on differentiating,

$$f^r = \frac{\delta v^r}{\delta t} = \dot{v} \lambda^r + v \frac{\delta \lambda^r}{\delta s} \frac{ds}{dt}.$$

Using (41) this becomes

$$f^r = \dot{v} \lambda^r + \kappa v^3 \mu^r. \quad \dots \quad (47)$$

It follows that in any trajectory *the acceleration vector is coplanar with the tangent and principal normal*. Its component along the tangent is \dot{v} and its component along the principal normal is κv^2 .

This result can be expressed in a slightly different form. Thus

$$v = \sqrt{2T}$$

and

$$\dot{v} = \frac{dv}{ds} \frac{ds}{dt} = v \frac{dv}{ds} = \frac{1}{2} \frac{d}{ds} v^2 = \frac{dT}{ds}.$$

Therefore

$$f^r = \frac{dT}{ds} \lambda^r + 2\kappa T \mu^r. \quad \dots \quad (48)$$

If we denote by f_r the covariant components of the acceleration vector we have

$$\begin{aligned} f_r &= a_{rs} f^s = a_{rs} \ddot{q}^s + [st, r] \dot{q}^s \dot{q}^t \\ &= \frac{d}{dt} (a_{rs} \dot{q}^s) - \frac{1}{2} \frac{\partial a_{st}}{\partial q^r} \dot{q}^s \dot{q}^t, \end{aligned}$$

that is,

$$f_r = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^r} - \frac{\partial T}{\partial q^r}. \quad \dots \quad (49)$$

Now if the trajectory is a natural trajectory for the force system Q_r the dynamical system is moving in accordance with the laws of dynamics and the equations of motion (34), p. 244, read

$$f_r = Q_r,$$

or, in contravariant form,

$$f^r = Q^r. \quad \dots \quad (50)$$

Hence in a natural trajectory the force vector and the acceleration vector coincide.

The student should remark the resemblance of these results to those obtained for the motion of a particle in the previous chapter.

EXAMPLES.

1. Prove that

$$\frac{ds}{dt} = \sqrt{2T}.$$

2. Show that in a natural trajectory the force vector must be coplanar with the tangent and principal normal.

3. Prove that the rate of change of kinetic energy is given by

$$\frac{dT}{dt} = f_r \dot{q}^r$$

and deduce that the equation of energy for a natural motion is

$$T = \int Q_r dq^r.$$

4. If the force system is conservative show that the equation of energy becomes $T + V = h$, where h is a constant.

5. Show that natural motion can take place along a line of force if and only if the line of force is a geodesic.

[A line of force is a curve whose tangent vector is codirectional at each point with Q^r and a geodesic of the manifold is a curve whose curvature κ is zero.]

9. The principle of stationary action. The action line-element.

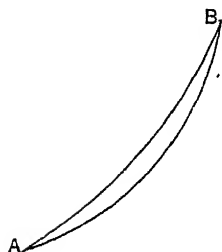
We have seen that for every natural trajectory under a conservative force system there is an equation of energy

$$T + V = h, \quad \dots \quad (51)$$

h being the constant of energy. In the present section we shall consider only trajectories for which the total energy is h , so that h remains a given constant throughout. Also if we have two configurations A and B there is in general only one natural trajectory, with the given energy constant, which passes through both A and B .

Let us consider all the curves of the manifold of configurations that pass through the points A and B corresponding to the two given configurations. We may take a parameter λ along each of these curves such that as λ varies from a value λ_0 to a value λ_1 the representative point moves from A to B . Then the integral

$$A = \int_A^B \left\{ 2(h - V) a_{rs} \frac{dq^r}{d\lambda} \frac{dq^s}{d\lambda} \right\}^{\frac{1}{2}} d\lambda \quad (52)$$



has a definite value for each curve through A and B . It is called the *action* from A to B for that curve. We have the following result, which is known as the *principle of stationary action*:

Of all curves passing through A and B that one for which the action is stationary is the natural trajectory of the system under the force system V with h as the constant of energy.

The proof of this principle is exactly the same as that given in the case of the motion of a particle in § 5, p. 228, and we refer the student to that section. The principle of stationary action for a particle is of course only a particular case of the principle for any dynamical system.

Now instead of taking the elementary distance ds between the points q^r and $q^r + dq^r$ as given by (37), p. 246, let us define it by the equation

$$ds^2 = 2(h - V) a_{rs} dq^r dq^s, \quad \quad (53)$$

which we shall refer to as the *action line-element* in order to distinguish it from the kinematical line-element already mentioned. We note that ds^2 is still a homogeneous quadratic form in dq^r .

With this new line-element the action (52) becomes

$$A = \int_A^B ds = [s]_A^B,$$

so that the action from A to B is the *length* of the curve from A to B . Thus the curves of stationary action become the curves of stationary length. But these latter are the *geodesics* of our manifold.

Hence the natural trajectories with a given energy constant h are the geodesics of the manifold of configurations when the action line-element is taken.

EXAMPLES.

1. If the kinematical line-element be used show that the action along a curve equals

$$\int_A^B \sqrt{2(h - V)} ds.$$

2. Show that the action, if calculated along a natural trajectory, equals

$$2 \int_A^B T dt.$$

EXAMPLES XVIII. (A).

Rectilinear Coordinates.

1. Show that the planes through a point P with respect to which the moments of inertia equal Mk^2 all touch the tangent cone from P to the quadric whose tangential equation is

$$(I^{mn} - Mk^2 g^{mn}) u_m u_n - 2M \xi^m u_m + M = 0.$$

2. Taking the origin at the centre of gravity show that the equation of the momental ellipsoid at a point α^r is

$$\{(I + M\alpha^2) g_{mn} - I_{mn} - M\alpha_m \alpha_n\} \alpha^m \alpha^n = 1,$$

where $\alpha^2 = g_{mn} \alpha^m \alpha^n$ and $\alpha_r = g_{rm} \alpha^m$.

3. Show that the component of linear momentum along any line λ^r is $G^r \lambda_r$.

4. Show that the moment of momentum round an axis whose six coordinates are λ^r, μ_r is $H_r \lambda^r - G^r \mu_r$.

[For the six coordinates of a line see p. 71.]

5. Show that the sum of the moments of a system of forces round the line λ^r, μ_r is $L_r \lambda^r - X^r \mu_r$.

6. If H_r^0 is the moment of momentum relative to a point x_0^r and L_r^0 is the moment vector of the force system relative to the same point, show that we have the vector equation

$$\frac{dH_r^0}{dt} = L_r^0$$

in the following cases:

- (i) if the point x_0^r is fixed,
 (ii) if the centre of gravity is fixed,
 (iii) if the given point coincides with or is moving in the same direction as the centre of gravity.

[Use the equations of Ex. 4, p. 237.]

7. If v^r is the velocity of the point x^r of a moving rigid body show that

$$\frac{\partial v^r}{\partial x^r} = 0.$$

[We have $v^r = u^r + \omega^r{}_s x^s$, where $\omega^r{}_s$ is skew-symmetric.]

8. If A^r is a vector fixed in a rigid body which is moving round a fixed point show that its components \bar{A}^r referred to axes fixed in space satisfy the relations

$$\frac{d\bar{A}^r}{dt} = \omega^r{}_s \bar{A}^s.$$

9. If the points of a rigid body have all two simultaneous velocities due to two angular velocity vectors $\omega^r_{(1)}$ and $\omega^r_{(2)}$, show that the resultant velocities are given by an angular velocity vector ω^r where $\omega^r = \omega^r_{(1)} + \omega^r_{(2)}$; that is, *angular velocities can be compounded as vectors*.

[For the velocities of the points of the rigid body in each case are given by $v^r_{(1)} = \varepsilon^{rst} \omega_{(1)s} x_t$ and $v^r_{(2)} = \varepsilon^{rst} \omega_{(2)s} x_t$. Thus the combined velocity is $v^r = v^r_{(1)} + v^r_{(2)} = \varepsilon^{rst} (\omega_{(1)s} + \omega_{(2)s}) x_t$.]

10. Show that the kinetic energy of a rigid body with one point fixed is

$$\begin{aligned} T &= \frac{1}{2} (I \omega^2 - I_{mn} \omega^m \omega^n) = \frac{1}{2} (I g_{mn} - I_{mn}) \omega^m \omega^n. \\ [T &= \frac{1}{2} \sum M_{(a)} g_{mn} \dot{x}^m_{(a)} \dot{x}^n_{(a)} = \frac{1}{2} \sum M_{(a)} g_{mn} \omega^m{}_s \omega^n{}_t x^s_{(a)} x^t_{(a)} \\ &= \frac{1}{2} I^{st} g^{mn} \varepsilon_{m s p} \varepsilon_{n t q} \omega^p \omega^q = \frac{1}{2} I^t{}_s \delta^{s p}_t \omega_p \omega^q = \text{etc.}] \end{aligned}$$

11. If the axes are the principal axes of inertia at the fixed point show that the kinetic energy becomes

$$T = \frac{1}{2} \{I_1(\omega^1)^2 + I_2(\omega^2)^2 + I_3(\omega^3)^2\},$$

where I_1, I_2, I_3 are the three principal moments of inertia.

12. Show that $\frac{\partial T}{\partial \omega^r}$ is the moment of momentum vector H_r .

13. Show that the kinetic energy of a rigid body is $T = \frac{1}{2} M V^2 + T_0$, where V is the velocity of the centre of gravity and T_0 is the kinetic energy relative to the centre of gravity (that is, it is calculated as if the centre of gravity were at rest).

EXAMPLES XVIII. (B).

Generalised Coordinates.

1. Show that the kinetic energy of a rigid body with one point fixed, expressed in terms of the principal moments of inertia and Euler's angles, is given by

$$\begin{aligned} 2T &= I_1 (\dot{\theta} \sin \psi - \dot{\phi} \sin \theta \cos \psi)^2 + I_2 (\dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi)^2 \\ &\quad + I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2. \end{aligned}$$

2. A symmetrical top with a point O on the axis fixed is acted upon by gravity, find the Lagrangian function L in terms of the Eulerian angles.

[$L = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_1 \dot{\phi}^2 \sin^2 \theta + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2 - Mgh \cos \theta$, where I_1, I_1, I_3 are the principal moments of inertia, M is the mass of the top and h is the distance from O to the centre of gravity.]

3. If K is an invariant function of q^r and \dot{q}^r , show that $\frac{\partial K}{\partial \dot{q}^r}$ and $\frac{d}{dt} \frac{\partial K}{\partial \dot{q}^r} - \frac{\partial K}{\partial q^r}$ are both covariant vectors.

4. *Hamilton's equations.* If we put $p_r = a_{rs} \dot{q}^s$, we call p_r the *momentum vector*. Show that the equations of motion can be written in the form

$$= \frac{\partial H}{\partial p_r}, \quad \frac{dp_r}{dt} + \frac{\partial H}{\partial q^r} = 0,$$

where

$$H = \frac{1}{2} a^{rs} p_r p_s + V.$$

5. Deduce from the equations of Ex. 4 that $\frac{dH}{dt} = 0$, that is, these equations have a first integral $H = \text{const.}$ Show that this integral is the equation of energy.

6. If H is an invariant function of q^r and of a covariant vector p_r , show that $\frac{\partial H}{\partial p_r}$ is a contravariant vector. Also if we take the curve defined in terms of a parameter u by means of the equations $\frac{dq^r}{du} = \frac{\partial H}{\partial p_r}$, show that $\frac{dp_r}{du} + \frac{\partial H}{\partial q^r}$ is a covariant vector.

[These results show that *Hamilton's equations are vector equations.*]

7. In a natural motion show that the rate of increase of the kinetic energy equals the product of the magnitude of the velocity vector, the magnitude of the force vector and the cosine of the angle between these two vectors.

[The cosine of the angle between two unit vectors λ^r, μ^r is defined as $\cos \theta = a_{rs} \lambda^r \mu^s$. Our result expressed analytically is that $\frac{dT}{dt} = Q_r v^r$.]

8. Show that along a natural trajectory $\frac{dT}{ds} = Q \cos \varphi$, where Q is the magnitude of the force vector and φ is the angle which this vector makes with the trajectory.

9. *Bonnet's theorem.* If a dynamical system can pass through a certain sequence of configurations under the separate influences of a number of force systems, show that when all the force systems act together the system can pass through the same sequence of configurations with a kinetic energy equal to the sum of the kinetic energies it had in the separate motions under the several force systems.

[From the geometrical view-point the proof is exactly the same as in the case of a particle.]

10. Prove that the curvature κ of a trajectory is given by the formula

$$\kappa^2 = \frac{f^2 - \dot{v}^2}{v^4} = \frac{f^2}{4T^2} - \frac{\dot{T}^2}{8T^3}.$$

[Use (47), p. 248.]

11. If φ is the angle between the acceleration and velocity vectors, show that the curvature of the trajectory is

$$\kappa = \frac{f \sin \varphi}{2T}.$$

[$\cos \varphi = \dot{T}/f\sqrt{2T}$, or, $\dot{T}^2 = 2f^2T \cos^2 \varphi$. Substitute this result in Ex. 10.]

12. If no force system is acting show that the representative point describes a geodesic in the manifold of configurations with the kinematical line-element.

[A geodesic is a curve whose κ is zero. In all the above questions the kinematical line-element is used.]

13. If a dynamical system is subjected to a constraint represented by an equation $\psi(q^1, q^2, q^3) = 0$, show that the equations of motion are

$$\ddot{r} = Q_r + \theta \psi_r$$

where $\psi_r = \frac{\partial \psi}{\partial q^r}$ and θ is an arbitrary multiplier.

14. *Impulsive motion.* Show that the equations of impulsive motion are

$$\left(\frac{\partial T}{\partial \dot{q}^r}\right)' - \frac{\partial T}{\partial q^r} = Q_r,$$

where $Q_r \delta q^r$ is the elementary work done by the *impulsive* forces under a virtual displacement δq^r .

Deduce that the change in the velocity vector is

$$\Delta \dot{q}^r = a^{rs} Q_s.$$

15. If a dynamical system is suddenly subjected to a constraint $\psi(q^1, q^2, q^3) = 0$, which persists, show that the change in the velocity vector is

$$\Delta \dot{q}^r = \theta a^{rs} \psi_s$$

where $\psi_r = \frac{\partial \psi}{\partial q^r}$ and

$$\theta = -(\psi_r \dot{q}^r) / (a^{rs} \psi_r \psi_s).$$

16. A dynamical system moves under the action of a force system of potential V , the energy constant being h . Show that, if the kinematical line-element be used, the time along the trajectory is given by

$$t = \int \frac{ds}{\sqrt{2(h - V)}}.$$

17. In the manifold of configurations show that of all trajectories passing through two given configurations A and B and satisfying the energy equation $T + V = h$ that which makes the integral

$$t = \int_A^B \left\{ \frac{a_{mn} q'^m q'^n}{2(h-V)} \right\}^{\frac{1}{2}} d\lambda, \quad q'^r = \frac{dq^r}{d\lambda},$$

stationary satisfies the equations

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}^r} - \frac{\partial T}{\partial q^r} - \frac{\partial V}{\partial q^r} + h - V \frac{dV}{dt} \frac{\partial T}{\partial \dot{q}^r} = 0.$$

[The trajectories which make the time of passage stationary are called *brachistochrones*, so that the above are the equations of the brachistochrones of our dynamical system.]

18. Show that the equations of Ex. 17 can be put in the form

$$\frac{d}{dt} \frac{\partial T_1}{\partial \dot{q}^r} - \frac{\partial T_1}{\partial q^r} = 0,$$

where $T_1 = T/(h-V)$. Hence deduce that the brachistochrones of the dynamical system coincide with the natural paths of a dynamical system whose kinetic energy is $T/(h-V)$ and which is moving under no forces.

19. Show that the brachistochrones of the dynamical system are geodesics in the manifold of configurations if the line-element is given by

$$ds^2 = \frac{a_{mn} dq^m dq^n}{(h-V)}.$$

20. If a dynamical system starts from rest in a position O , show that the natural trajectory and the line of force through O have the same tangent vectors and the same principal normals at O . Also show that the curvature of the first curve is one-third the curvature of the second at O .

CHAPTER XIX

ELECTRICITY AND MAGNETISM

1. Green's theorem.

Before discussing the applications of the tensor calculus to the mathematical theory of electricity and magnetism we shall deal with two important theorems on the transformation of volume integrals into surface integrals and of surface integrals into line integrals.

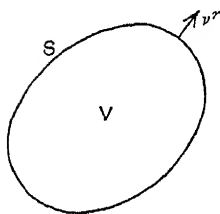
Let us suppose that we refer space to a system of rectangular Cartesian axes (x, y, z) . The well-known theorem of Green is expressed in such coordinates as follows:

If S is a closed surface bounding a volume V , and P, Q, R are three functions, uniform, continuous and admitting partial derivatives of the first order throughout V , then

$$\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) d\tau = \iint_S (lP + mQ + nR) d\sigma, \quad (1)$$

where l, m, n are the direction cosines of the outward normal of S , $d\tau$ the element of volume and $d\sigma$ the surface element on S .

The result is easily proved by integrating partially along lines parallel to the coordinate axes.



We shall take a general curvilinear coordinate system x^r and express Green's theorem in these coordinates. We know that a vector may be defined by taking as its components in one system of coordinates any three quantities and taking its components in all other systems to satisfy the transformation law of vectors.

Let us define a vector F^r such that its components in the (x, y, z) -coordinates are given by P, Q, R . This defines a vector-field, since the vector is a function of the coordinates. Now if $F^r_{\cdot, s}$ denotes the covariant derivative of F^r , that is, if

$$F^r_{\cdot, s} = \frac{\partial F^r}{\partial x^s} + \left\{ \begin{matrix} r \\ p s \end{matrix} \right\} F^p,$$

then $F^r_{\cdot, r}$ is an invariant. Hence its value in the (x, y, z) -system is

$$F^r_{\cdot, r} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}, \quad \dots \dots (2)$$

for the Christoffel symbols vanish in a Cartesian coordinate system. This invariant $F^r_{\cdot, r}$ is often referred to as *the divergence of F^r* . Also, denoting by v^r the unit outward normal vector to S , its components in the Cartesian axes are l, m, n and consequently the invariant $F^r v_r$ equals

$$g_{x^a} F^p v^a = Pl + Qm + Rn. \quad \dots \dots (3)$$

Green's theorem therefore reads in tensor form

$$\iiint_V F^r_{\cdot, r} d\tau = \iint_S F^r v_r d\sigma. \quad \dots \dots (4)$$

The integral on the right of this equation is often called *the flux of the vector F^r across the surface S* . Thus the flux of F^r across S equals the volume integral of the divergence of F throughout V . The theorem can be extended to regions lying between several surfaces and the volume integral can be extended to infinity provided F^r satisfies certain restrictive conditions at infinity.

We can express Green's theorem in another form. Let φ and ψ be two invariant functions and let us put for brevity

$$\varphi_r = \frac{\partial \varphi}{\partial x^r}, \quad \psi_r = \frac{\partial \psi}{\partial x^r}.$$

If we place

$$F_r = g_{rs} F^s = \varphi \psi_r, \quad (5)$$

then

$$F_{r,r} = g^{rs} F_{r,s} = g^{rs} (\varphi \psi_{r,s} + \varphi_s \psi_r),$$

where $\psi_{r,s}$ is the covariant derivative of ψ_r . The invariant $g^{rs} \psi_{r,s}$ is often denoted by $\Delta \psi$ and is called *the Laplacian of ψ* . Its value in the Cartesian coordinates is

$$\Delta \psi = g^{rs} \psi_{r,s} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}. \quad . . . (6)$$

Also $g^{rs} \varphi_r \psi_s$ is an invariant and is denoted by $\nabla(\varphi, \psi)$. Its value in the (x, y, z) -axes is

$$\nabla(\varphi, \psi) = g^{rs} \varphi_r \psi_s = \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial z} \frac{\partial \psi}{\partial z}. \quad . . (7)$$

Thus

$$F_{r,r} = \varphi \Delta \psi + \nabla(\varphi, \psi) \quad (8)$$

and Green's theorem now reads

$$\iiint_V \nabla(\varphi, \psi) d\tau = \iint_S \varphi \psi_r \nu^r d\sigma - \iiint_V \varphi \Delta \psi d\tau. \quad . (9)$$

EXAMPLES.

1. Show that

$$F_{r,r} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r} (\sqrt{g} F^r), \quad \Delta \varphi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r} \left(\sqrt{g} g^{rs} \frac{\partial \varphi}{\partial x^s} \right).$$

[See p. 155.]

2. Calculate $\Delta \varphi$ in spherical polar and cylindrical polar coordinates.

[Use Ex. 1.]

3. Prove that

$$\iint_S \psi_r \nu^r d\sigma = \iiint_V \Delta \psi d\tau.$$

[Put $\varphi = 1$ in (9).]

4. If V contains a surface Σ on which F^r is discontinuous show that

$$\iint_S F^r \nu_r d\sigma = \iiint_V F_{r,r} d\tau - \iint_\Sigma \{ (F^r \nu_r)_1 + (F^r \nu_r)_2 \} d\sigma,$$

where the suffixes 1 and 2 refer to the two sides of Σ .

2. Stokes's theorem.

The second important theorem deals with the transformation of a line integral into a surface integral. It is known as Stokes's theorem and in rectangular Cartesian coordinates reads:

If S be a portion of a surface enclosed by a contour C and if P, Q, R be three functions continuous and possessing partial derivatives of the first order over S , then

$$P \frac{dx}{ds} + Q \frac{dy}{ds} + R \frac{dz}{ds} ds = \iint_S \left\{ l \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + m \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + n \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right\} d\sigma. \quad (10)$$

where the line integral is taken round the whole circuit C and l, m, n are the direction cosines of the normal to S .

We shall now find the form of this theorem in curvilinear coordinates x^r . Let us again take the vector F^r , whose components in the (x, y, z) -coordinates are P, Q, R . The components of the associated vector F_r in the same Cartesian coordinates are also P, Q, R since there is no distinction in rectangular Cartesian axes between the covariant and contravariant components of a vector.

The covariant derivative $F_{r,s}$ of F_r is a double tensor and consequently

$$G^r \equiv -\varepsilon^{rst} F_{s,t} \quad \dots \dots \dots (11)$$

is a contravariant vector, where ε^{rst} is the contravariant ε -tensor, that is, ε^{rst} equals $+1/\sqrt{g}$ or $-1/\sqrt{g}$ according as r, s, t is an even or odd permutation of 1, 2, 3 and vanishes for all other values of r, s, t . The vector (11) is usually called *the curl or rotation of F_r* and we see that its components in the rectangular Cartesian system (x, y, z) are

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Hence if we denote by ν^r the unit normal vector to S we see that $G^r \nu_r$ is an invariant and its value in the (x, y, z) -coordinates is given by

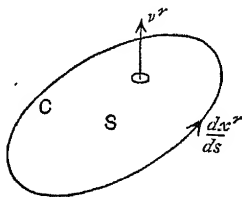
$$G^r \nu_r = \varepsilon^{rst} F_{s,t} \nu_r = l \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + m \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + n \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

Also $\frac{dx^r}{ds}$ is the unit tangent vector to the curve C and consequently $F_r \frac{dx^r}{ds}$ is an invariant whose value in the Cartesian axes is

$$F_r \frac{dx^r}{ds} = P \frac{dx}{ds} + Q \frac{dy}{ds} + R \frac{dz}{ds}.$$

Stokes's theorem therefore reads in tensor form

$$\int_C F_r \frac{dx^r}{ds} ds = - \iint_S \varepsilon^{rst} F_{s,t} v_r d\sigma. \quad (12)$$



The integral on the left of this equation is often called *the circulation of the vector F_r round the circuit C* .

EXAMPLES.

1. Write down the components of the curl of a vector whose components are P, Q, R (i) in spherical polar coordinates (ii) in cylindrical polar coordinates, (iii) in oblique Cartesian coordinates (cf. Ex. 4, p. 155).

2. If we introduce curvilinear coordinates u^1, u^2 on the surface S , (cf. p. 163), show that

$$\varepsilon^{rst} F_{s,t} v_r = \varepsilon^{\alpha\beta} F_{\alpha,\beta}$$

where $F_{\alpha,\beta}$ is the surface covariant derivative of the surface vector

$$F_\alpha = F_r \frac{\partial x^r}{\partial u^\alpha}.$$

[Use the relation $v_r = \frac{1}{2} \varepsilon^{\alpha\beta} \varepsilon_{rst} \frac{\partial x^s}{\partial u^\alpha} \frac{\partial x^t}{\partial u^\beta}$, proved in § 4, p. 196.

We obtain

$$\begin{aligned} \varepsilon^{rst} F_{s,t} v_r &= \varepsilon^{\alpha\beta} F_{s,t} \frac{\partial x^s}{\partial u^\alpha} \frac{\partial x^t}{\partial u^\beta} = \varepsilon^{\alpha\beta} \frac{\partial}{\partial u^\beta} \left(F_s \frac{\partial x^s}{\partial u^\alpha} \right) = \varepsilon^{\alpha\beta} \left(\frac{\partial F_\alpha}{\partial u^\beta} \right) \\ &= \varepsilon^{\alpha\beta} F_{\alpha,\beta}. \end{aligned}$$

3. From Ex. 2 deduce that Stokes's theorem may be written in the form

$$\int_C F_\alpha \frac{du^\alpha}{ds} ds = - \iint_S \varepsilon^{\alpha\beta} F_{\alpha,\beta} d\sigma.$$

3. The electrostatic field.

We know that the electric field produced at a given point by a number of electric charges in a vacuum is given by a potential function V where

$$V = \frac{1}{4\pi} \sum \frac{e}{r} \quad (13)$$

when the charges are isolated, e being the electric charge measured in theoretical units* and r the distance from the given point to the charge e . If the charges are uniformly distributed over volumes and surfaces the expression for V becomes

$$V = \frac{1}{4\pi} \iiint_V \frac{\rho d\tau}{r} + \frac{1}{4\pi} \iint_S \frac{\mu d\sigma}{r}, \dots \quad (14)$$

where ρ is the volume density of charge and μ the surface density of charge. The density ρ will of course be zero at any point where no volume distribution of charges exists and similarly μ will vanish at any point where no surface distribution exists.

We shall denote the electric force vector by E_r , so that

$$E_r = -\frac{\partial V}{\partial x^r}, \dots \quad (15)$$

the coordinate system being a general curvilinear one x^r .

Gauss's theorem states that the flux of electric force across any surface S equals the total electric charge inside the surface. Thus

$$\iint_S E_r v^r d\sigma = \iiint_V \rho d\tau + \iint_\Sigma \mu d\sigma,$$

where Σ is any surface inside S having a surface distribution of charge upon it. But it is a simple deduction from Green's theorem that

$$\iint_S E_r v^r d\sigma = \iiint_V E_{r,r} d\tau - \iint_\Sigma \{(E_r v^r)_1 + (E_r v^r)_2\} d\sigma,$$

where the suffixes 1 and 2 refer to opposite sides of the surface Σ . Hence

$$\iiint_V (E_{r,r} - \rho) d\tau - \iint_\Sigma \{(E_r v^r)_1 + (E_r v^r)_2 + \mu\} d\sigma = 0$$

and this result is true for all surfaces S . Consequently we have at each point of space

$$g^{rs} E_{r,s} = E_{r,r} = \rho, \dots \quad (16)$$

ρ being zero where no electric charge exists. Moreover at the surface Σ on which there is a surface distribution of density μ we have

$$(E_r v^r)_1 + (E_r v^r)_2 + \mu = 0. \dots \quad (17)$$

These are the fundamental equations of the electrostatic field.

* Theoretical or rational units are used throughout this chapter.

EXAMPLES.

1. Show that $E^r_{\cdot, r} = 0$ at any point where no charge exists.
2. Show that the first of equations (16) may be written $\Delta V = -\varrho$, where $\Delta V = g^{mn} V_{m, n}$ and Δ is the Laplacian of V .
3. *Lines of force.* A line of force is a curve whose tangent at each point is in the direction of the electric force vector. Show that the equations of a line of force are

$$dx^r = k E^r,$$

where $1/k^2 = g_{mn} E^m E^n$.

4. Show that the lines of force are orthogonal to the surfaces $V = \text{const.}$ These surfaces are called *equi-potential surfaces*.

4. Dielectrics.

If there is a material medium in which the electric charges exist, it is found that the equations of the last section have to be modified. The electric field is now determined by two vectors:

(i) *the force vector* E_r , which is minus the gradient of a potential function V , that is,

$$E_r = -\frac{\partial V}{\partial x^r} = -V_r, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

(ii) a vector D_r , which is called *the displacement vector*.

The vector D_r is such that its flux across any surface equals the total charge inside the surface. Thus we prove by Green's theorem exactly as on p. 260 that

$$g^{rs} D_{r, s} = D^r_{\cdot, r} = \varrho \quad . \quad . \quad . \quad . \quad . \quad (19)$$

throughout space, ϱ being zero where no electric charge exists. Also at the surface Σ on which there is a surface distribution of density μ we have

$$(D_r v^r)_1 + (D_r v^r)_2 + \mu = 0, \quad . \quad . \quad . \quad (20)$$

where the suffixes 1 and 2 refer to opposite sides of Σ .

The vectors E_r and D_r are so related that when one is given the other is also given, that is, one is a function of the other. The simplest relation between them consistent with experiment is that one vector is a linear function of the other. This gives us

$$D_r = \epsilon_r^s E_s \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where the ε_s^r are functions of the coordinates alone. The relation (21) shows that ε_s^r is a mixed double tensor and we call it *the dielectric tensor*.

Let us examine the conditions that the dielectric be *homogeneous*. This means that if the same electric force exists at two different points then the displacement vector is also the same at the two points. Expressed in other words, if the electric force vector is a constant parallel vector-field so also is the displacement vector. The condition that E_r is a constant parallel vector-field is expressed by $E_{r,s} = 0$, $E_{r,s}$ being the covariant derivative, with similar results for D_r . Differentiating (21) covariantly with respect to x^t , we get

$$D_{r,t} = \varepsilon_r^s E_{s,t} + \varepsilon_{r,t}^s E_s$$

and consequently the condition for homogeneity is

$$\varepsilon_{s,t}^r = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

that is, the covariant derivative of ε_s^r must vanish at each point.

If the dielectric is *isotropic* the displacement vector and the force vector must coincide in direction and this involves the relation

$$\varepsilon_s^r = \varepsilon \delta_s^r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where ε is an invariant and δ_s^r is a Kronecker delta. If the medium is both homogeneous and isotropic, the relations (22) and (23) show us that ε is a constant throughout the medium. In this case it is called *the dielectric constant*.

The equations (18) to (21) are the fundamental equations of the electric field in anisotropic dielectrics, but there is another vector sometimes used to describe the properties of the dielectric. This is the vector

$$P_r = D_r - E_r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

which is called *the polarisation vector*. We see that

$$P_r = (\varepsilon_r^s - \delta_r^s) E_s, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

so that the components of the polarisation vector can be expressed linearly in terms of the components of the electric force vector. The tensor $(\varepsilon_s^r - \delta_s^r)$ may be called *the susceptibility tensor*.

EXAMPLES

1. Show that

$$g^{rs} D_{r,s} = g^{rs} \epsilon_{r,s}^t E_t + \epsilon^{rs} E_{r,s}$$

where ϵ^{rs} is the associated dielectric tensor $g^{sm} \epsilon_m^r$.

2. Deduce from Ex. 1 that the potential function
- V
- satisfies the equation

$$\epsilon_{r,s}^{r,s} V_r + \epsilon^{rs} V_{r,s} = -\varrho.$$

3. If the dielectric is homogeneous we have

$$\epsilon^{rs} V_{r,s} = -\varrho.$$

4. Show that in a homogeneous isotropic medium

$$\epsilon \Delta V = \epsilon g^{mn} V_{m,n} = -\varrho$$

and at a boundary of two such dielectrics

$$(\epsilon E_r v^r)_1 + (\epsilon E_r v^r)_2 = -\mu,$$

μ being the surface charge on the boundary.

5. The magnetostatic field.

The potential at a given point due to an elementary magnet at a point P is given by

$$\Omega = \frac{1}{4\pi} I^m \frac{\partial}{\partial x^m} \left(\frac{1}{r} \right)$$

where the vector I^r is called the magnetic moment vector and r is the distance from the given point to P . Thus the potential of any volume distribution of magnetism will be

$$\Omega = \frac{1}{4\pi} \int_V I^m \frac{\partial}{\partial x^m} \left(\frac{1}{r} \right) d\tau, \quad (26)$$

$I^r d\tau$ being the magnetic moment vector of the magnetism occupying the element of volume $d\tau$. We call I^r the *magnetisation vector*. Equation (26) becomes by Green's theorem

$$\Omega = \frac{1}{4\pi} \int_V I_m v^m \frac{d\sigma}{r} - \frac{1}{4\pi} \int_S I_{m,n}^m \frac{d\tau}{r}, \quad \dots \quad (27)$$

which shows that the potential may be regarded as being due to magnetic matter of volume density ϱ and surface density μ where

$$\begin{aligned} \varrho &= -I_{m,n}^m, \\ \mu &= I_m v^m. \end{aligned} \quad \dots \quad (28)$$

The magnetic force vector H_r outside the magnetic matter is given by

$$H_r = -\frac{\partial \Omega}{\partial x^r} = -\Omega_r. \quad (29)$$

The magnetic force vector inside the magnetic matter is found to be the resultant of the vectors H_r and I_r at the point, where H_r is defined by the equations (27) and (29). This vector B_r , which is called *the magnetic induction vector*, is therefore given by

$$B_r = H_r + I_r. \quad (30)$$

It follows from (29) and (27) that

$$g^{mn} H_{m,n} = -g^{mn} \Omega_{m,n} = -\Delta \Omega = -g^{mn} I_{m,n}.$$

Hence, from (30)

$$g^{mn} B_{m,n} = 0, \quad (31)$$

showing that the divergence of the magnetic induction vector is zero.

Some of the magnetism in our field may be permanent, and the magnetic moment density of permanent magnetism we shall denote by I_r^0 . The remainder is induced by the magnetic field and consequently is a function of the magnetic force vector H_r . This function we shall take to be a linear function and we have therefore the total intensity of magnetisation given by

$$I_r = I_r^0 + \lambda_r^s H_s.$$

The tensor λ_r^s is called *the magnetic susceptibility tensor*. From this we have the result

$$B_r = I_r^0 + \mu_r^s H_s, \quad (32)$$

where

$$\mu_r^s = \lambda_r^s + \delta_r^s.$$

The tensor μ_r^s is called *the magnetic permeability tensor*.

The equations (29), (30), (31) and (32) are the principal equations of the magnetostatic field. Of course if there is no permanent magnetism we shall have $I_r^0 = 0$.

EXAMPLES.

1. Show that

$$g^{rs} B_{r,s} = g^{rs} I_{r,s}^0 + g^{rs} \mu_{r,s}^t H_t + \mu^{rs} H_{r,s}$$

where μ^{rs} is the associated permeability tensor $g^{rs} \mu_{rs}$.

2. Deduce from Ex. 1 that the potential function Ω satisfies the equation

$$\mu_{r,s}^{rs} \Omega_r + \mu^{rs} \Omega_{r,s} = g^{rs} I_{r,s}^0$$

3. If the medium be homogeneous show that

$$\mu_{s,t}^r = 0,$$

and the potential function satisfies the equation

$$\mu^{mn} \Omega_{m,n} = g^{mn} I_{m,n}^0.$$

4. If the medium is isotropic show that

$$\mu_s^r = \mu \delta_s^r.$$

5. If the medium is both isotropic and homogeneous show that Ω satisfies the equation

$$\mu \Delta \Omega = g^{mn} I_{m,n}^0,$$

where μ is a constant.

6. The electromagnetic equations.

In a conductor the electric current is represented by a vector C^r , which we may call *the current vector*. This vector is such that the amount of electricity flowing through an element of area $d\sigma$, which is perpendicular to the unit vector λ^r , is measured by $C^r \lambda_r d\sigma$.

Ohm's law states that this current vector is a linear function of the electric force vector E_r . Thus

$$C^r = \kappa^{rs} E_s, \quad \dots \quad (33)$$

the κ 's being functions of the coordinates only. We see that κ^{rs} is a double tensor and we call it *the conductivity tensor*. If the medium is homogeneous we see as before that

$$\kappa_{\dots,t}^{r,s} = 0,$$

and if the medium is isotropic

$$\kappa^{rs} = \kappa g^{rs}.$$

The electric current in a conductor is sometimes called *the current of conduction* to distinguish it from other kinds of electric currents which we shall meet later.

Let us suppose that we have a medium at rest, which may consist of dielectrics and conductors and in which electric charges may be moving. We wish to find the electromagnetic equations under such conditions.

The electromagnetic field is determined by the following vectors:

- (i) The electric force vector E_r .
- (ii) The displacement vector D_r . With these two vectors there is also associated the vector $P_r = D_r - E_r$, which is called the polarisation vector.
- (iii) The magnetic induction vector B_r , which satisfies the equation $g^{mn}B_{m,n} = 0$.
- (iv) The induced magnetisation vector of the medium I_r . This vector with B_r defines a new vector H_r equal to $B_r - I_r$, which is called the magnetic force vector.
- (v) The *total* current vector C^r .

The total current vector C^r contains three different types of current vectors:

- (a) The *conduction* current vector which is connected with E_r by Ohm's law. Thus the conduction current vector is

$$\kappa^{rs} E_s,$$

κ^{rs} being the conductivity tensor of the medium.

- (b) The *displacement* current vector which is given by

$$\frac{\partial D^r}{\partial t}.$$

- (c) The *convection* current vector, which is due to the motion of the electric charges in the medium. Thus if an electric charge, volume density ρ , has a velocity v^r then the convection current vector is

$$\rho v^r.$$

The first fundamental law of electromagnetism is *Faraday's law*, which states that the electromotive force induced in a circuit is proportional to the rate of diminution of the flux of magnetic induction through the circuit. Now the electromotive force in a circuit L is measured by the contour integral

$$\int_L E_r \frac{dx^r}{ds} ds$$

taken round the circuit. Faraday's law therefore reads

$$\int_L E_r \frac{dx^r}{ds} ds = - \frac{1}{c} \frac{\partial}{\partial t} \iint_S B_r v^r d\sigma, \quad \dots \quad (34)$$

where S is a surface passing through L . Applying Stokes's theorem to the contour integral, we have

$$\iint_S \left\{ \frac{1}{c} \frac{\partial B^r}{\partial t} - \varepsilon^{rst} E_{s,t} \right\} \nu_r d\sigma = 0.$$

This equation is true for all surfaces S and consequently

$$\frac{1}{c} \frac{\partial B^r}{\partial t} = \varepsilon^{rst} E_{s,t}, \quad (35)$$

or, expressed in words, the rate of change of B^r equals minus c times the curl of E_r . This is the first vector equation of the electromagnetic field.

The second fundamental law is *Ampère's law*, which states that the integral of the magnetic force vector round a closed circuit is proportional to the flux of current through the circuit. This law, expressed in symbols, gives

$$\int_L H_r \frac{dx^r}{ds} ds = \frac{1}{c} \iint_S C^r \nu_r d\sigma. \quad (36)$$

When we apply Stokes' theorem as before we get

$$\frac{1}{c} C^r = -\varepsilon^{rst} H_{s,t}, \quad (37)$$

that is, the current vector equals c times the curl of H_r . This is the second vector equation of the field.

To the two vector equations (35) and (37) we add the two invariant equations

$$\left. \begin{aligned} g^{mn} D_{m,n} &= \varrho, \\ g^{mn} B_{m,n} &= 0, \end{aligned} \right\} \quad (38)$$

which we have already seen to hold in statical fields (pp. 261—264). The equations (35), (37) and (38) complete the electromagnetic equations.

The vectors D_r , E_r and B_r , H_r are of course connected by means of the formulae

$$\left. \begin{aligned} D_r &= \varepsilon_r^s E_s, \\ B_r &= \mu_r^s H_s. \end{aligned} \right\} \quad (39)$$

EXAMPLES.

1. Show that

$$\varepsilon^{rst} E_{s,t} = \frac{1}{c} \mu^{rs} \frac{\partial H_s}{\partial t}.$$

$$\left[\text{Since the medium is at rest } \frac{\partial \mu_s^r}{\partial t} = \frac{\partial \varepsilon_s^r}{\partial t} = 0. \right]$$

2. Show that

$$\epsilon^{rst} H_{s,t} = -\frac{1}{c} \left\{ \kappa^{rs} E_s + \epsilon^{rs} \frac{\partial E_s}{\partial t} + \varrho v^r \right\}.$$

3. If the medium is homogeneous and isotropic show that the electromagnetic equations become

$$\begin{aligned} \epsilon g^{mn} E_{m,n} &= \varrho, \\ g^{mn} H_{m,n} &= 0, \\ \epsilon^{rst} E_{s,t} &= \frac{\mu}{c} \frac{\partial H^r}{\partial t}, \\ \epsilon^{rst} H_{s,t} &= -\frac{1}{c} \left\{ \kappa E^r + \epsilon \frac{\partial E^r}{\partial t} + \varrho v^r \right\}. \end{aligned}$$

4. Show that in the electromagnetic field

$$\epsilon^{rst} B_{s,t} = -\frac{1}{c} \{ C^r - c \epsilon^{rst} I_{s,t} \}.$$

5. Show that the electromagnetic equations may be written in the form

$$\begin{aligned} E_{r,s} - E_{s,r} &= \frac{1}{c} \epsilon_{prs} \frac{\partial B^p}{\partial t}, & H_{r,s} - H_{s,r} &= -\frac{1}{c} \epsilon_{prs} C^p, \\ D^r_{,r} &= \varrho, & B^r_{,r} &= 0. \end{aligned}$$

EXAMPLES XIX.

1. If F_r is a vector-field such that the curl of F_r is identically zero, show by Stokes's theorem that there exists an invariant function φ such that

$$F_r = \frac{\partial \varphi}{\partial x^r}$$

2. Show by Green's theorem that if the integral $\iint_S F^r \nu_r d\sigma$, taken over a surface S enclosed by a contour C , depends only on C then $F^r_{,r} = 0$, that is, the divergence of F^r is zero.

3. If the divergence of F^r is everywhere zero prove that we can find a vector A_r such that F^r is the curl of A_r , and show that F^r is also the curl of $\left(A_r + \frac{\partial \varphi}{\partial x^r} \right)$, where φ is an arbitrary function of position.

4. Show that the energy of the electrostatic field is

$$W = \frac{1}{2} \iiint_V \varrho V d\tau + \frac{1}{2} \iint_S \mu V d\sigma.$$

5. Deduce from Ex. 4 that W can be expressed by the integral

$$W = \frac{1}{2} \iiint g_{mn} E^m E^n d\tau,$$

taken over the whole of space.

6. Show that the energy of the electric field in a dielectric is

$$\begin{aligned} W &= \frac{1}{2} \iiint \epsilon_{mn} E^m E^n d\tau. \\ [W &= \frac{1}{2} \iiint_V \rho V d\tau + \frac{1}{2} \iint_S \mu V d\sigma \\ &= \frac{1}{2} \iiint V D_{,r} d\tau - \frac{1}{2} \iint_S V \{ (D_r v^r)_1 + (D_r v^r)_2 \} d\sigma \\ &= \frac{1}{2} \iiint E^m D_m d\tau, \text{ by Green's theorem.}] \end{aligned}$$

7. Show that in an anisotropic dielectric there exist three systems of lines of force such that at each point the corresponding lines of displacement coincide with them. We may call these lines *the principal lines of the dielectric*.

[The lines of displacement are those for which $dx^r = \theta D^r$. A principal direction λ^r of the dielectric is such that $(\epsilon_{rs} - \theta g_{rs})\lambda^s = 0$ where θ is a root of the determinantal equation $|\epsilon_{rs} - \theta g_{rs}| = 0$.]

8. In the magnetostatic field show that a vector F_r can be chosen such that B^r equals the curl of F_r , that is, $B^r = -\epsilon^{rst} F_{s,t}$. The vector F_r is called *the vector potential*.

9. Show that in any medium there are three orthogonal directions at each point such that if the lines of magnetic force through the point coincides with one of these directions so will B_r and I_r .

[These directions λ^r satisfy the equations $(\mu^r_s - \theta \delta^r_s)\lambda^s = 0$ where θ is a root of the equation $|\mu^r_s - \theta \delta^r_s| = 0$.]

10. Show that for the magnetic field

$$\iiint B_r H^r d\tau = 0,$$

the integral being taken throughout space and deduce that

$$\iiint I^0_r H^r d\tau + \iiint \mu_{,s} H^r H^s d\tau = 0.$$

11. Show that the energy of the magnetic field is

$$W = \frac{1}{2} \iiint \mu_{,s} H^r H^s d\tau,$$

the integral being taken throughout space, and show that for a homogeneous isotropic medium this becomes

$$W = \frac{1}{2} \mu \iiint g_{,s} H^r H^s d\tau.$$

12. Show that at any point of a surface Σ in the magnetostatic field

$$(B_r v^r)_1 + (B_r v^r)_2 = 0$$

where v^r_1 and v^r_2 are the unit normals on the two sides of Σ .

13. The energy of the electromagnetic field is taken to be the sum of the energies of the electric and magnetic fields. Show that the energy density w of the electromagnetic field is

$$w = \frac{1}{2} (\epsilon_{rs} E^r E^s + \mu_{,s} H^r H^s).$$

14. If S is any fixed surface enclosing a volume V , show that

$$\frac{\partial}{\partial t} \iiint_V w d\tau = c \iint_S \epsilon^{rst} H_s E_t v_r d\sigma - \iiint_V (\kappa^{rs} E_r E_s + \rho v^r E_r) d\tau.$$

the vector $S^r = c \epsilon^{rst} H_s E_t$ is called *the Poynting energy flux vector*.

15. Show that the divergence of O^r is zero, that is, $O^r_{,r} = 0$.

16. Prove that, in a homogeneous isotropic medium which has no convection currents and which is a non-conductor,

$$\frac{\varepsilon \mu}{c^2} \frac{\partial^2 E_r}{\partial t^2} = g^{mn} E_{r,mn} - E^{nn}_{,mr}.$$

If there is no electric charge at the point this equation becomes

$$\frac{\varepsilon \mu}{c^2} \frac{\partial^2 E_r}{\partial t^2} = g^{mn} E_{r,mn}.$$

[Here $\kappa^{rs} = 0$, $v^r = 0$, $\varepsilon_{rs} = \varepsilon g_{rs}$ and $\mu_{rs} = \mu g_{rs}$.]

17. Prove that the invariant function $\varphi = E_r \lambda^r$, where E_r is the electric force vector in a homogeneous isotropic dielectric and λ^r is a constant parallel vector system, satisfies the equation

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{c^2}{\mu \varepsilon} \Delta \varphi.$$

[Use fact that $\lambda^r_{,s} = 0$ and the second result of Ex. 16.]

18. In the electron theory we assume that there is a vacuum outside the electrons. Show that in this theory the electromagnetic equations are

$$H^r_{,r} = 0, \quad E^r_{,r} = \varrho, \quad \frac{1}{c} \left(\frac{\partial E^r}{\partial t} + \varrho v^r \right) = -\varepsilon^{rst} H_{s,t}, \quad \frac{1}{c} \frac{\partial H^r}{\partial t} = \varepsilon^{rst} E_{s,t},$$

and ϱ is zero outside the electrons.

[Here $\varepsilon_{rs} = g_{rs} = \mu_{rs}$, $\kappa^{rs} = 0$.]

19. The electromagnetic equations for an isotropic conducting medium at rest are

$$E^r_{,r} = \varrho/\varepsilon, \quad H^r_{,r} = 0, \\ \frac{1}{c} \left(\kappa E^r + \varepsilon \frac{\partial E^r}{\partial t} \right) = -\varepsilon^{rst} H_{s,t}, \quad \frac{\mu}{c} \frac{\partial H^r}{\partial t} = \varepsilon^{rst} E_{s,t}.$$

Show that

$$\frac{\partial \varrho}{\partial t} = -\frac{\kappa}{\varepsilon} \varrho.$$

Assuming $\varrho = 0$, prove that

$$\frac{\varepsilon \mu}{c^2} \frac{\partial^2 E_r}{\partial t^2} + \frac{\kappa \mu}{c^2} \frac{\partial E_r}{\partial t} = g^{mn} E_{r,mn}.$$

[We have $\frac{1}{c} \left(\kappa E^r_{,r} + \varepsilon \frac{\partial E^r_{,r}}{\partial t} \right) = -\varepsilon^{rst} H_{s,t} = 0$, since $H_{s,t}$ is symmetric in t and r . Substitute $E^r_{,r} = \varrho/\varepsilon$ and first result follows. Second result follows from the equation $\frac{\mu}{c} \frac{\partial H^r}{\partial t} = \varepsilon^{rst} E_{s,t}$.]

20. In an isotropic medium at rest prove that

$$\frac{\varepsilon \mu}{c^2} \frac{\partial^2 H_r}{\partial t^2} + \frac{\kappa \mu}{c^2} \frac{\partial H_r}{\partial t} = g^{mn} H_{r,mn}.$$

[The proof is similar to that used in Ex. 19.]

CHAPTER XX

MECHANICS OF CONTINUOUS MEDIA

1. Infinitesimal strain.

Let us consider a continuous medium, the points of which are referred to a given curvilinear coordinate system x^r . If each point of the medium is displaced slightly so as to take up a neighbouring position, the medium is said to have an *infinitesimal strain* or *deformation* and we shall now investigate the geometrical properties of such a strain.

Let P_0 be a point of the medium in the unstrained state and let P be its new position. Since the strain is small, the displacement P_0P is infinitesimal. We shall denote this small vector by ξ^r and we shall call it *the displacement vector*. Consequently when the strain is given there is defined an infinitesimal displacement vector at each point of the medium.

If the coordinates of P and P_0 are x^r and x_0^r respectively we see that $x^r - x_0^r$ are the components of the displacement vector, that is,

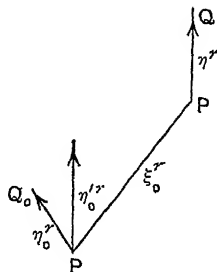
$$x^r - x_0^r = \xi^r(x_0^1, x_0^2, x_0^3), \quad . \quad . \quad . \quad (1)$$

and ξ^r is an infinitesimal of the first order. We wish to examine how the neighbourhood of the point P_0 has been deformed. Let Q_0 be a neighbouring point and Q its deformed position. If we denote the small vector P_0Q_0 by η_0^r and its deformed vector PQ by η^r , we see that the coordinates of Q and Q_0 are $x^r + \eta^r$ and $x_0^r + \eta_0^r$ respectively. Also the vector Q_0Q is the value of ξ^r at the point Q_0 and consequently

$$\begin{aligned} (x^r + \eta^r) - (x_0^r + \eta_0^r) &= \xi^r(x_0^1 + \eta_0^1, x_0^2 + \eta_0^2, x_0^3 + \eta_0^3) \\ &= \xi_0^r + \left(\frac{\partial \xi^r}{\partial x^s} \right)_0 \eta_0^s, \end{aligned}$$

where we neglect powers of η_0^r higher than the first. Using (1) we have

$$\eta^r - \eta_0^r = \left(\frac{\partial \xi^r}{\partial x^s} \right)_0 \eta_0^s. \quad . \quad . \quad . \quad (2)$$



In order to compare η^r with η_0^r let us take the vector $\eta_0'^r$ at P_0 which is *parallel* to η^r at P . Now if the axes were Cartesian the components of η^r and $\eta_0'^r$ would be equal and we would have

$$\eta_0'^r - \eta_0^r = \left(\frac{\partial \xi^r}{\partial x^s} \right)_0 \eta_0^s.$$

To obtain the corresponding relation in general curvilinear coordinates we have merely to introduce the covariant derivative of ξ^r , namely,

$$\xi^{r..s} = \frac{\partial \xi^r}{\partial x^s} + \left\{ \begin{matrix} r \\ s \ t \end{matrix} \right\} \xi^t.$$

Thus the equation

$$\eta_0'^r - \eta_0^r = (\xi^{r..s})_0 \eta_0^s \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is a tensor equation, which is true in any system of Cartesian axes. It is therefore true in all coordinate systems. The difference vector $\delta \eta_0^r = \eta_0'^r - \eta_0^r$ measures the strain of the vector η_0^r . Since we shall henceforward be dealing only with vectors at P_0 we may drop the zero suffix without confusion and write (3)

$$\delta \eta^r = \xi^{r..s} \eta^s \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The associated tensor $\xi_{r..s}$, where $\xi_{r..s} = g_{ri} \xi^{i..s}$, is not in general symmetric. Let us put

$$e_{r..s} = \frac{1}{2} (\xi_{r..s} + \xi_{s..r}), \quad \omega_{r..s} = \frac{1}{2} (\xi_{r..s} - \xi_{s..r}). \quad . \quad . \quad . \quad (5)$$

It follows that

$$\xi_{r..s} = e_{r..s} + \omega_{r..s},$$

or, raising the index r ,

$$\xi^{r..s} = e^{r..s} + \omega^{r..s}.$$

Therefore (4) reads

$$\delta \eta^r = e^{r..s} \eta^s + \omega^{r..s} \eta^s \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If we neglect quantities of order higher than ξ^r , we see that the two deformations

$$\delta \eta^r = e^{r..s} \eta^s \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$\delta \eta^r = \omega^{r..s} \eta^s \quad . \quad . \quad . \quad . \quad . \quad (8)$$

when performed successively have the same effect as (6) and the order in which they are taken is of no consequence. We call a strain of the type (7), $e_{r..s}$ being a symmetric tensor, a

pure strain and we shall see from the examples that (8) is a small rotation of the neighbourhood of P_0 round P_0 . Thus every infinitesimal strain is composed of

- (i) a pure strain,
- (ii) a rotation of the neighbourhood of P_0 round P_0 ,
- (iii) a translation of the neighbourhood of P_0 to P .

The last two transformations are rigid-body movements and we see that the deformation proper is determined by the first transformation. Thus a strain is determined in the neighbourhood of P_0 except for a rigid-body movement by the tensor e_{rs} and this tensor is called *the strain tensor*.

EXAMPLES.

1. Show that the dilatation is given by $\xi^r_{,r}$.

Let us take three small vectors $\eta^r_{(1)}$, $\eta^r_{(2)}$, $\eta^r_{(3)}$ at P_0 and let us denote their deformed positions by dashed letters. If ΔV is the element of volume of the tetrahedron whose edges are the three original vectors at P_0 and if $\Delta V'$ is the deformed volume, then $(\Delta V' - \Delta V)/\Delta V$ is called *the dilatation* at P_0 . We have

$$\Delta V = \frac{1}{6} \varepsilon_{rst} \eta^r_{(1)} \eta^s_{(2)} \eta^t_{(3)}, \quad \Delta V' = \frac{1}{6} \varepsilon_{rst} \eta'^r_{(1)} \eta'^s_{(2)} \eta'^t_{(3)}.$$

Thus, to the nearest order,

$$\begin{aligned} \Delta V' &= \Delta V + \frac{1}{6} \varepsilon_{rst} (\eta'^r_{(1)} \eta^s_{(2)} \delta \eta^t_{(3)} + \eta^r_{(1)} \delta \eta^s_{(2)} \eta^t_{(3)} + \delta \eta^r_{(1)} \eta^s_{(2)} \eta^t_{(3)}) \\ &= \Delta V + \frac{1}{6} \varepsilon_{rst} (\eta^r_{(1)} \eta^s_{(2)} \xi^t_{,p} \eta^p_{(3)} + \eta^r_{(1)} \xi^s_{,p} \eta^p_{(2)} \eta^t_{(3)} + \xi^r_{,p} \eta^p_{(1)} \eta^s_{(2)} \eta^t_{(3)}) \\ &= \Delta V + \frac{1}{6} \xi^r_{,r} \varepsilon_{mn p} \eta^m_{(1)} \eta^n_{(2)} \eta^p_{(3)}. \end{aligned}$$

Consequently the dilatation is given by

$$\frac{\Delta V' - \Delta V}{\Delta V} = \xi^r_{,r},$$

that is, *the dilatation equals the divergence of the displacement vector*.

2. Show that the dilatation is given by $g^{mn} e_{mn}$.

3. If η is the magnitude of the vector η^r and $\delta \eta$ is the change in the magnitude after deformation, show that

$$\eta \delta \eta = e_{rs} \eta^r \eta^s.$$

If λ^r is the unit vector in the direction of η^r , deduce that

$$e = \frac{\delta \eta}{\eta} = e_{rs} \lambda^r \lambda^s.$$

The quantity e is called *the elongation of the vector η^r* .

4. If we take the elementary quadric surface

$$e_{rs} \eta^r \eta^s = k,$$

where k is infinitesimal, show that the elongation of η^r equals k/η^2 . This quadric is called the *strain quadric*.

5. The principal axes of the strain quadric are called the *principal axes of strain*. Show that these principal axes are given by the equations

$$(e_{rs} - eg_{rs})\lambda^s = 0,$$

where e is a root of the equation $|e_{rs} - eg_{rs}| = 0$.

6. By taking suitable Cartesian axes at P_0 , show that the transformation (8) is an infinitesimal rotation round an axis λ^r of amount $\delta\theta$, where $\varepsilon^{rmn}\omega_{mn} = -2\delta\theta\lambda^r$.

[Take a Cartesian system \bar{x}^r in which $\bar{\lambda}^1 = \bar{\lambda}^2 = 0$, $\bar{\lambda}^3 = 1$.]

2. Analysis of stress.

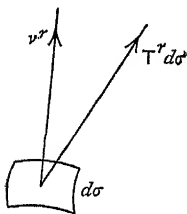
We shall next consider the forces acting on a portion of the continuous medium.

First of all, there may exist *external* body forces which act on every particle of the medium and, if F^r denotes the force vector per unit mass, then the force acting on the element of volume $d\tau$ is $F^r\rho d\tau$ where ρ is the density of the medium. In addition to these body forces there may be external surface forces acting on the external surface of the medium such that on the surface element $d\sigma$ there acts a force $T^rd\sigma$.

The *internal* forces of the medium are analysed as follows. On an element of surface $d\sigma$ in the medium the action of the matter on one side of the element upon that on the other side is represented by a force vector $T^rd\sigma$. We call one side of $d\sigma$ the *positive* side and the other the *negative* and we draw the unit normal vector v^r on the *positive* side. Also we shall let $T^rd\sigma$ be the action of the matter of the positive side upon that of the negative. Then, of course, the action of the negative side upon the positive will be $-T^rd\sigma$ by the principle of the equality of action and reaction. Now this force vector $T^rd\sigma$ depends not only on $d\sigma$ but also on the orientation of $d\sigma$, that is, on v^r . By taking Cartesian coordinates it is easy to show that T^r is a linear homogeneous function of v^r and we have

$$T^r = E^{rs}v_s, \quad \dots \dots \dots (9)$$

E^{rs} being *symmetric* in r and s and depending only on the coordinates at the point. We see that E^{rs} is a double contra-



variant tensor. The internal forces of the medium are called stresses and we call E^{rs} the stress tensor.

Let us now find the equations of motion of any portion of the medium. Let the volume V be cut off by the surface S inside the medium. The equations of motion of this portion of matter are

$$\iiint_V \rho (F^r - f^r) \lambda_r d\tau + \iint_S T^r \lambda_r d\sigma = 0,$$

where f^r is the acceleration vector and λ_r is any constant parallel vector-field.

If ν^r is the unit normal vector to S drawn outwards we know that

$$T^r = E^{rs} \nu_s.$$

Hence

$$\iiint_V \rho (F^r - f^r) \lambda_r d\tau + \iint_S E^{rs} \lambda_r \nu_s d\sigma = 0.$$

Using Green's theorem we see that

$$\iint_S E^{rs} \lambda_r \nu_s d\sigma = \iiint_V (E^{rs} \lambda_r)_{,s} d\tau = \iiint_V E^{r..s}_{,s} \lambda_r d\tau,$$

since $\lambda_{r,s} = 0$. Thus

$$\iiint_V \{ \rho (F^r - f^r) + E^{r..s}_{,s} \} \lambda_r d\tau = 0,$$

and this equation is true for any volume V and for any parallel vector-field λ_r . Consequently

$$E^{r..s}_{,s} + \rho F^r = \rho f^r \dots \dots \dots (10)$$

at each point of the medium. These are the equations of motion of the medium.

EXAMPLES.

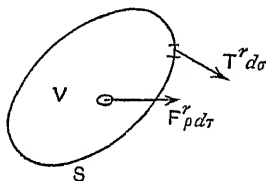
1. If we take the quadric $E_{rs} \eta^r \eta^s = k$, show that the stress on the element $d\sigma$ acts in the direction of the perpendicular to the plane conjugate to the normal of $d\sigma$ with respect to this quadric. The quadric is called the stress quadric.

2. Show that the normal stress on an element $d\sigma$ equals k/η^2 where η is the radius vector of the stress quadric in the direction of the normal to $d\sigma$.

3. The principal axes of the stress quadric are called the principal axes of stress. Show that these principal axes are given by

$$(E_{rs} - E g_{rs}) \lambda^s = 0,$$

where E is a root of the determinantal equation $|E_{rs} - E g_{rs}| = 0$.



4. Show that the stress on the surface element perpendicular to one of the principal axes of stress is at right angles to the surface element. Such a stress is called a *principal stress*.

3. Equations of motion for a perfect fluid.

In the case of a perfect fluid the stress on the element $d\sigma$ is always normal to $d\sigma$ and consequently

$$E_{rs} = -p g_{rs} \dots \dots \dots (11)$$

where p is an *invariant*. The quantity p is called the *pressure* of the fluid. We notice that the magnitude of the pressure on a surface element at a given point is independent of the orientation of the element.

The equations (10) may be written in the covariant form

$$g^{st} E_{rs,t} + \varrho F_r = \varrho f_r.$$

But

$$g^{st} E_{rs,t} = -g^{st} (p g_{rs}),_t = -g^{st} \frac{\partial p}{\partial x^t} g_{rs} = -\frac{\partial p}{\partial x^r}.$$

Hence the equations of motion of the fluid are

$$\frac{\partial p}{\partial x^r} = \varrho (F_r - f_r) \dots \dots \dots (12)$$

If v^r is the velocity vector of any point of the medium, v^r will be a function both of the time and of the coordinates of the point, that is,

$$v^r = v^r(x^1, x^2, x^3, t) \dots \dots \dots (13)$$

Therefore we have

$$f_r = \frac{\delta v^r}{\delta t} = \frac{\partial v^r}{\partial t} + v^r_{,s} v^s, \dots \dots \dots (14)$$

or, lowering the index r ,

$$f_r = \frac{\partial v_r}{\partial t} + v_{r,s} v^s, \dots \dots \dots (15)$$

and the equations of motion become

$$\frac{1}{\varrho} \frac{\partial p}{\partial x^r} = F_r - \frac{\partial v_r}{\partial t} - v_{r,s} v^s \dots \dots \dots (16)$$

Again the mass contained in a fixed volume V in the fluid is

$$M = \iiint_V \varrho d\tau,$$

and hence the rate of change of mass equals

$$\frac{\partial M}{\partial t} = \iiint_V \frac{\partial \varrho}{\partial t} d\tau.$$

But this rate of change of mass is also given by

$$\frac{\partial M}{\partial t} = - \iint_S \varrho v^r \nu_r d\sigma = - \iiint_V (\varrho v^r)_{,r} d\tau,$$

where we have made use of Green's theorem. Equating the two values of $\frac{\partial M}{\partial t}$ we get

$$\iiint_V \left\{ \frac{\partial \varrho}{\partial t} + (\varrho v^r)_{,r} \right\} d\tau = 0.$$

This equation is true for any volume V in the fluid and consequently

$$\frac{\partial \varrho}{\partial t} + (\varrho v^r)_{,r} = 0. \quad (17)$$

The equation (17) is called *the equation of continuity*. In addition there is the *characteristic* equation of the fluid

$$K(p, \varrho) = 0, \quad (18)$$

which gives the connection between the pressure and density.

The vector Ω^r , defined by

$$2\Omega^r = -\varepsilon^{rmn} v_{m,n}, \quad (19)$$

we call *the vortex vector* and the curves given by the equations

$$\frac{dx^r}{ds} = \frac{\Omega^r}{\Omega},$$

Ω being the magnitude of Ω^r , are called the *vortex lines*.

EXAMPLES.

1. Show that $\Omega^r_{,r} = 0$.

2. Show that the equations of motion of a perfect fluid can be written in the form

$$\frac{1}{\varrho} \frac{\partial p}{\partial x^r} = F_r - \frac{\partial v_r}{\partial t} + 2\varepsilon_{rmn} v^m \Omega^n - \frac{1}{2} \frac{\partial v^2}{\partial x^r},$$

where v^2 is the square of the magnitude of v^r , that is, $v^2 = g_{mn} v^m v^n$.

3. The fluid is *incompressible* when the density at a given point of the fluid is invariable. Show that for an incompressible fluid

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x^r} v^r = 0$$

and deduce that the equation of continuity becomes $v^r_{,r} = 0$.

4. If a homogeneous liquid is acted upon by a force system derived from a potential function V , show that the equations of motion may be put into the form

$$\frac{\partial v_r}{\partial t} - 2 \varepsilon_{r m n} v^m \Omega^n = - \frac{\partial}{\partial x^r} \left(\rho + V + \frac{1}{2} v^2 \right).$$

[For a homogeneous liquid ρ is a constant.]

5. If $\Omega^r = 0$, the motion is said to be *irrotational*. Show that the necessary and sufficient condition is that there exists a function φ such that $v_r = - \frac{\partial \varphi}{\partial x^r}$, and show that the equations of motion may now be written

$$\frac{\partial \varphi}{\partial t} = \frac{p}{\rho} + V + \frac{1}{2} v^2 + C.$$

The function φ is the *velocity potential*.

4. The equations of elasticity.

We have seen that the equations of motion of an elastic body are

$$g^{st} E_{r s, t} + \rho f_r = \rho f_r,$$

E_{rs} being the stress tensor and f_r the acceleration vector. Now in a small strain the point originally at x^r has taken up the position $x^r + \xi^r$. Also we shall consider only small motions of the elastic medium. Thus the acceleration f_r is given to the required degree of approximation by

$$f_r = \frac{\partial^2 \xi_r}{\partial t^2},$$

and the equations of motion become

$$g^{st} E_{r s, t} + \rho f_r = \rho \frac{\partial^2 \xi_r}{\partial t^2}. \quad \dots \dots (20)$$

The stress in an elastic medium depends upon the strain existing in the medium and vanishes when the strain disappears. In other words, the stress tensor is a function of the strain tensor. Moreover *Hooke's law* states that the stress is a linear function of the strain. We may therefore assume

$$E_{rs} = c_{rs}^{mn} e_{mn}, \quad \dots \dots (21)$$

where the c 's are functions of the coordinates alone. They are called *the elastic coefficients* and we see that they form a mixed tensor of the fourth order. It is also clear that without loss of generality these coefficients can be taken to be symmetric both in the subscripts and superscripts, that is,

$$c_{rs}^{mn} = c_{rs}^{nm} = c_{sr}^{mn} = c_{sr}^{nm}.$$

When we substitute for E_{rs} in the equations of motion, we get

$$g^{st} (c_{rs}^{mn} e_{mn})_{,t} + \varrho F_r = \varrho \frac{\partial^2 \xi_r}{\partial t^2},$$

or

$$g^{st} c_{rs}^{mn} e_{mn,t} + g^{st} c_{rs,t}^{mn} e_{mn} + \varrho F_r = \varrho \frac{\partial^2 \xi_r}{\partial t^2}. \quad (22)$$

If the body is *homogeneous*, the same strain at different points of the medium produces the same stress. This is equivalent to saying that the stress tensor forms a constant parallel tensor-field whenever the strain tensor does, that is, $E_{rs,t} = 0$ if $e_{rs,t} = 0$. The necessary and sufficient conditions that a medium be elastically homogeneous are therefore

$$c_{rs,t}^{mn} = 0 \quad \dots \dots \dots (23)$$

and the equations of motion of such a medium become

$$g^{st} c_{rs}^{mn} e_{mn,t} + \varrho F_r = \varrho \frac{\partial^2 \xi_r}{\partial t^2}.$$

When the medium is *isotropic*, the relations between stress and strain are found to be

$$E_{rs} = \lambda \theta g_{rs} + 2 \mu e_{rs}, \quad \dots \dots \dots (24)$$

λ, μ being elastic constants and θ being the cubical dilatation

$$\theta = \xi_{,r}^r = g^{mn} e_{mn}. \quad \dots \dots \dots (25)$$

The constants λ, μ are invariants and we see from (24) and (25) that for isotropic media the elastic constants are of the form

$$c_{rs}^{mn} = \lambda g^{mn} g_{rs} + \mu (\sigma_r^m \sigma_s^n + \sigma_s^m \sigma_r^n). \quad \dots \dots (26)$$

In the case of isotropic bodies we can now write down the equations of motion in terms of the displacement vector as follows. We have

$$g^{st} E_{rs,t} = \lambda \frac{\partial \theta}{\partial x^r} + 2 \mu g^{st} e_{rs,t}.$$

Also

$$\begin{aligned} g^{st} e_{rs,t} &= \frac{1}{2} g^{st} (\xi_{r,st} + \xi_{s,rt}) = \frac{1}{2} g^{st} (\xi_{r,st} + \xi_{s,tr}) \\ &= \frac{1}{2} g^{st} \xi_{r,st} + \frac{1}{2} \frac{\partial \theta}{\partial x^r}. \end{aligned}$$

Therefore the equations of motion are

$$(\lambda + \mu) \frac{\partial \theta}{\partial x^r} + \mu g^{st} \xi_{r,st} + \varrho E_r = \varrho \frac{\partial^2 \xi_r}{\partial t^2} \dots \quad (27)$$

EXAMPLES.

1. Show that in an isotropic body $g^{rs} E_{rs} = (3\lambda + 2\mu)\theta$.

2. *Energy.* Show that the work done during a small variation δe_{mn} in the strain is given by

$$\delta W = \iiint E^{mn} \delta e_{mn} d\tau,$$

the integral being taken over the elastic body. Hence deduce that the work stored up in the strain is

$$W = \frac{1}{2} \iiint E^{mn} e_{mn} d\tau = \frac{1}{2} \iiint c_{rs}^{mn} e_{mn} e^{rs} d\tau.$$

3. In the case of an isotropic body show that the energy of the strained body is W where

$$2W = \iiint \{ \lambda (g^{mn} e_{mn})^2 + 2\mu g^{mr} g^{ns} e_{mn} e_{rs} \} d\tau.$$

5. The motion of a viscous fluid.

The equations of motion of any deformable system we have seen (p. 276) to be of the form

$$g^{st} E_{rs,t} + \varrho (F_r - f_r) = 0,$$

where the acceleration vector f_r is given in terms of the velocity v^r by the formula

$$f_r = \frac{\partial v^r}{\partial t} + v_{r,s} v^s.$$

If the medium is a perfect fluid we have seen that

$$E_{rs} = -p g_{rs},$$

where p is the pressure and equals $-\frac{1}{3} g^{mn} E_{mn}$. If the medium is not a perfect fluid, we shall put

$$E_{rs} + p g_{rs} = E'_{rs} \dots \dots \dots (28)$$

and we shall call E'_{rs} the *viscosity tensor*. Thus the equations of motion are

$$-\frac{\partial p}{\partial x^r} + g^{st} E'_{rs,t} + \varrho F_r = \varrho f_r. \quad (29)$$

The deformation of the fluid in a time dt is given by the displacement vector

$$\xi^r = v^r dt,$$

and consequently the strain components are

$$e_{rs} = \frac{1}{2} (v_{r,s} + v_{s,r}) dt.$$

Let us call e_{rs}/dt the *velocity strain tensor* and let us denote it by e'_{rs} , so that

$$e'_{rs} = \frac{1}{2} (v_{r,s} + v_{s,r}). \quad (30)$$

We shall also put as before

$$2\Omega^r = -\varepsilon^{rmn} v_{m,n}. \quad (31)$$

We assume that the viscosity tensor is a linear function of the velocity strain tensor, and thus

$$E'_{rs} = \gamma^{mn}_{rs} e'_{mn}. \quad (32)$$

The quantities γ^{mn}_{rs} , which form a mixed tensor of the fourth order, are called the *coefficients of viscosity*. We see that we may take them as being symmetric both in the subscripts and in the superscripts. Consequently

$$E'_{rs} = \gamma^{mn}_{rs} v_{m,n}$$

and the equations of motion are

$$-\frac{\partial p}{\partial x^r} + g^{st} (\gamma^{mn}_{rs,t} v_{m,n} + \gamma^{mn}_{r,s} v_{m,n,t}) + \varrho F_r = \varrho f_r. \quad (33)$$

If the fluid is homogeneous we have in addition $\gamma^{mn}_{rs,t} = 0$.

In the case of an isotropic fluid we have the relations

$$\gamma^{mn}_{rs} = \lambda' g^{mn} g_{rs} + \mu' (\delta^m_r \delta^n_s + \delta^m_s \delta^n_r). \quad (34)$$

and the equations (33) for a homogeneous isotropic fluid become

$$\frac{\partial p}{\partial x^r} - (\lambda' + \mu') \frac{\partial \theta'}{\partial x^r} - \mu' g^{st} v_{r,st} = \varrho (F_r - f_r). \quad (35)$$

Here

$$\theta' = v^r_{,r}. \quad (36)$$

and the equation of continuity still reads

$$\frac{\partial \varrho}{\partial t} + (\varrho v^r)_{,r} = 0.$$

EXAMPLES.

1. Show that $g^{mn} E'_{mn}$ vanishes and hence deduce that

$$g^{mn} \gamma_{mn}^{r,s} = 0.$$

[$3p = -g^{mn} E_{mn}$ and use (28).]

2. Deduce from Ex. 1 that if the fluid is isotropic $\lambda' = -\frac{2}{3}\mu'$.

3. Prove that the equations of motion of an isotropic fluid are

$$\frac{\partial p}{\partial x^r} - \frac{1}{3}\mu' \frac{\partial \theta'}{\partial x^r} - \mu' g^{st} v_{r,st} = \varrho (F_r - f_r).$$

EXAMPLES XX.

1. Show that in a pure strain the change $\delta \eta^r$ is a small vector orthogonal to the plane which is conjugate to η^r with respect to the strain quadric.

2. Show that the condition that a strain be pure is that there exists a function φ such that $\xi_r = \frac{\partial \varphi}{\partial x^r}$. The function φ is the strain potential.

3. If a medium is homogeneous and incompressible, show that $g^{mn} e_{mn} = 0$. If, in addition, it is subjected to a pure strain, prove that the strain potential satisfies Laplace's equation $\Delta \varphi = 0$.

4. Show that the strain components e_{rs} satisfy the identical relations

$$e_{rs,tu} + e_{tu,rs} = e_{rt,su} + e_{su,rt}.$$

[Use the fact that $e_{rs,t} - e_{rt,s} = \omega_{st,r}$ and, differentiating covariantly with respect to u , use the fact that $\omega_{st,ru}$ is symmetric in r and u .]

5. The integral $\int_C v_r \frac{dx^r}{du} du$ taken round a closed circuit is called the circulation round this curve. Prove that in the motion of a perfect fluid

$$\frac{D}{Dt} \int_C v_r \frac{dx^r}{du} du = \int_C F_r dx^r - \int_C \frac{1}{\varrho} \frac{dp}{du} du.$$

where $\frac{D}{Dt}$ means differentiation following the fluid.

$$\begin{aligned} \left[\frac{Dx^r}{Dt} = v^r \text{ and hence } \frac{D}{Dt} \int_C v_r \frac{dx^r}{du} du = \int_C \left(f_r \frac{dx^r}{du} + v_r \frac{\delta v^r}{\delta u} \right) du \right. \\ \left. = \int_C f_r \frac{dx^r}{du} du. \right] \end{aligned}$$

6. Show that

$$\int_C v_r dx^r = 2 \iint_S \Omega_r v^r d\sigma,$$

where S is a surface in a perfect fluid, enclosed by a contour C .

7. Prove that, if the forces acting on a perfect fluid are conservative and ϱ is a function of p ,

$$\frac{D}{Dt} \left(\frac{\Omega^r}{\varrho} \right) = \frac{1}{\varrho} \Omega^s v^r_{,s}.$$

$$\begin{aligned} \frac{D \Omega^r}{Dt} &= \frac{\partial \Omega^r}{\partial t} + \Omega^r_{,s} v^s = -\frac{1}{2} \varepsilon^{rmn} \frac{\partial v_{m,n}}{\partial t} + \Omega^r_{,s} v^s \\ &= -\varepsilon^{rmn} (e_{mpq} v^p \Omega^q)_{,n} + \Omega^r_{,s} v^s = -v^s_{,s} \Omega^r + \Omega^s v^r_{,s}. \end{aligned}$$

Now use the equation of continuity, namely, $\frac{D\varrho}{Dt} + \varrho v^m_{,m} = 0$.

8. We denote by $2J^r$ the *curl* of f_r , that is, $2J^r = -\varepsilon^{rmn} f_{m,n}$. Prove that as the surface S moves with the fluid

$$\frac{D}{Dt} \left\{ \int_S \Omega_r v^r d\sigma \right\} = \left| \int_S J_r v^r d\sigma \right|.$$

[We have $\frac{D}{Dt} \int_C v_r dx^r = \int_C f_r dx^r$. Transform each line integral into a surface integral by means of Stokes's theorem.]

9. If the force system is conservative and ϱ is a function of p , prove that $J^r = 0$.

10. If the displacement vector ξ_r of a deformed isotropic elastic medium is derived from a potential function φ show that the equations of motion become

$$(\lambda + 2\mu) \frac{\partial \theta}{\partial x^r} + \varrho F_r = \varrho f_r,$$

where the dilatation θ equals $\Delta \varphi$.

11. In Ex. 10, if no body forces are acting and the body is in equilibrium, show that $\Delta \varphi = k$, where k is a constant.

12. *Types of strain*: Deduce the following expressions for the strain tensor.

(i) *Uniform dilatation*. $e_{rs} = e g_{rs}$.

(ii) *Simple extension*. $e_{rs} = e \lambda_r \lambda_s$, λ_r being the unit vector in the given direction.

(iii) *Shearing strain*. $e_{rs} = e(\lambda_r \lambda'_s + \lambda'_r \lambda_s)$, where λ_r, λ'_r are the unit vectors of the given directions.

13. Show that, if the strain be expressed in terms of the stress by means of the equations

$$e_{rs} = C_{rs}^{mn} E_{mn},$$

then the C 's and c 's are connected by the formulae

$$c_{rs}^{mn} C_{mn}^p = \frac{1}{2} (\delta_r^p \delta_s^q + \delta_r^q \delta_s^p) = c_{mn}^p C_{rs}^{mn}.$$

14. Show that for an isotropic medium

$$C_{rs}^{pq} = \frac{1}{4\mu} (\delta_r^p \delta_s^q + \delta_r^q \delta_s^p) - \frac{\lambda}{2\mu(3\lambda + 2\mu)} g^{pq} g_{rs}.$$

15. If we apply a uniform spherical tension E to the elastic body, show that the ratio of E to the cubical dilatation equals

$$C_{rs}^{mn} g_{mn} g^{rs}.$$

This ratio is called *the modulus of compression*.

[Hint: $E_{rs} = E g_{rs}$]

16. If a stress $E(\lambda_r \lambda_s + \lambda'_r \lambda'_s)$ is applied to an elastic body, λ_r and λ'_r being orthogonal unit vectors, the ratio of E to $2e_{rs} \lambda^r \lambda'^s$ is called the *rigidity* corresponding to the two given directions λ^r, λ'^r . Show that the rigidity is

$$\frac{C_{rs}^{mn} \lambda^r \lambda'^s \lambda_m \lambda'_n}{4}.$$

17. If a stress $E \lambda_r \lambda_s$ is applied to an elastic body, the ratio of E to the invariant $e_{rs} \lambda^r \lambda^s$ is called *Young's modulus* corresponding to the direction λ^r . Show that Young's modulus is

$$\frac{C_{rs}^{mn} \lambda_m \lambda_n \lambda^r \lambda^s}{e_{rs} \lambda^r \lambda^s}.$$

Also show that the ratio of the contraction in any direction λ'^r , perpendicular to λ^r , to the elongation $e_{rs} \lambda^r \lambda^s$ is

$$-\frac{C_{rs}^{mn} \lambda_m \lambda_n \lambda'^r \lambda'^s}{e_{rs} \lambda_m \lambda_n \lambda^r \lambda^s}.$$

This is called *Poisson's ratio* for the directions λ^r, λ'^r .

18. If we have an isotropic elastic body on which no body forces are acting, show that the following relations connect the stress and strain tensors in the position of equilibrium:

$$\begin{aligned} & \text{(i) } \Delta \theta = 0. \quad \text{(ii) } \Delta (g^{mn} E_{mn}) = 0. \quad \text{(iii) } (3\lambda + 2\mu) g^{mn} E_{rs, mn} \\ & + 2(\lambda + \mu) g^{mn} E_{mn, rs} = 0. \quad \text{(iv) } (\lambda + \mu) \theta_{, rs} + \mu g^{mn} e_{rs, mn} = 0. \\ & \text{(v) } (\lambda + 2\mu) \theta_{, r} + 2\mu g^{st} \omega_{rs, t} = 0. \quad \text{(vi) } g^{mn} g^{pq} \xi_{r, mn pq} = 0. \end{aligned}$$

19. Show that the stress in a viscous fluid is

$$E_{rs} = \mu' (v_{r, s} + v_{s, r}) - g_{rs} (p + \frac{2}{3} \mu' v^m{}_{, m}).$$

20. If the motion of a viscous fluid is slow, prove that the equations of motion can be written approximately

$$\rho \frac{\partial v_r}{\partial t} = \rho E_r - \frac{\partial p}{\partial x^r} + (\lambda' + \mu') \frac{\partial \theta'}{\partial x^r} + \mu' g^{mn} v_{r, mn}.$$

If, in addition, the fluid is incompressible, $\theta' = 0$.

21. In the slow steady motion of a homogeneous viscous liquid, show that

$$g^{mn} v_{r, mn} = \frac{1}{\mu'} \left(\frac{\partial p}{\partial x^r} - \varrho F_r \right)$$

and deduce that

$$\Delta p = g^{mn} p_{,mn} = \varrho F_{,r},$$

that is, the divergence of $\left(\frac{\partial p}{\partial x^r} - \varrho F_r \right)$ is zero.

[Use the fact that $\theta' = g^{rs} v_{r,s} = 0$ and as a consequence

$$g^{rs} v_{r, mns} = g^{rs} v_{r, smn} = 0.]$$

CHAPTER XXI

THE SPECIAL THEORY OF RELATIVITY

1. The four-dimensional manifold.

We shall now consider the applications of tensor methods to the theory of relativity. We confine ourselves to the special theory since all the treatises on relativity already deal with the applications of tensors to the general theory. The student will indeed find that the equations at which we arrive can in many cases be immediately generalised to meet the requirements of the general theory.

Let us suppose that the positions in space of certain events are recorded with reference to an orthogonal Cartesian system of axes (x, y, z) , that is, the position of any event will be given by three numbers or space-coordinates. Moreover the observer who is using this coordinate system of axes also measures by means of a clock the times at which the events occupy the observed positions. Obviously then an event is completely recorded by the four quantities x, y, z, t . Up to the present we have found a geometrical representation of the event by plotting the point (x, y, z) in space and attaching to it as a scalar the time at which the event took place. But another form of geometrical interpretation is open to us. We may take a four-dimensional manifold, that is, a manifold in which it requires four numbers or coordinates to fix the position of a point, and we shall take x, y, z, t as coordinates in this space. Consequently we see that in our four-dimensional manifold every event is represented by a point and, conversely, every point represents an event. We shall see that this representation of events is essential for the development of the ideas of this chapter. We call this manifold the manifold of space and time and we shall often refer to it briefly as space-time.

2. Generalised coordinates in space-time.

If we are given four variables x^1, x^2, x^3, x^4 , which are known functions of the space coordinates x, y, z and the time t , we shall obviously have four relations of the form

$$\left. \begin{aligned} x^1 &= f^1(x, y, z, t), \\ x^2 &= f^2(x, y, z, t), \\ x^3 &= f^3(x, y, z, t), \\ x^4 &= f^4(x, y, z, t). \end{aligned} \right\} \dots \dots \dots (1)$$

We shall suppose that these relations are reversible, that is, the x, y, z, t variables can be expressed uniquely in terms of x^1, x^2, x^3, x^4 . As a result of these relations it is seen that any event can be expressed in terms of the four variables x^1, x^2, x^3, x^4 , which determine it uniquely. These variables, which we shall call *coordinates*, are a direct generalisation to four dimensions of curvilinear coordinates in space.

We shall use *Greek* indices to distinguish these variables and in this chapter we shall adopt the following conventions:

A Greek index, if free, is to have the range of values 1 to 4, and a Greek index, if repeated, is to be summed from 1 to 4. Consequently the formulae (1) can be written briefly

$$x^\alpha = f^\alpha(x, y, z, t).$$

On the other hand we shall keep our previous conventions that *italic* indices will range from 1 to 3.

Any other observer, who is observing the same events, will similarly be able to record these events in terms of four other variables \bar{x}^α , and these must be uniquely related to the x^α since each set of variables records the same events. We have therefore the relations

$$\bar{x}^\alpha = \bar{x}^\alpha(x^1, x^2, x^3, x^4) \dots \dots \dots (2)$$

together with their inverses

$$x^\alpha = x^\alpha(\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4) \dots \dots \dots (3)$$

In other words, the transformation from one set of coordinates to another set is a functional transformation between the variables.

Exactly as in the case of transformations in three variables we can define *tensors* with respect to functional transformations in four variables and the definitions can be immediately extended. Thus the set of quantities $A_{\beta\gamma}^{\alpha}$ is a *tensor of the third order*, contravariant in α and covariant in β and γ , if its components $\bar{A}_{\beta\gamma}^{\alpha}$ in the new coordinate system satisfy the relations

$$\bar{A}_{\beta\gamma}^{\alpha} = A_{\alpha\tau}^{\rho} \frac{\partial \bar{x}^{\alpha}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial \bar{x}^{\beta}} \frac{\partial x^{\tau}}{\partial \bar{x}^{\gamma}} \cdot \dots \dots \dots (4)$$

We may call these tensors *world tensors* to distinguish them from the ordinary space tensors.

Let (x, y, z, t) be the space coordinates and the time used by an observer S and let $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ be the corresponding variables used by another observer \bar{S} (the origin of space coordinates in each case coinciding with the observer). These are particular cases of generalised coordinates in space-time and we have

$$\begin{aligned} \bar{x} &= \bar{x}(x, y, z, t), & \bar{y} &= \bar{y}(x, y, z, t), \\ \bar{z} &= \bar{z}(x, y, z, t), & \bar{t} &= \bar{t}(x, y, z, t), \end{aligned}$$

and also the inverse equations

$$x = x(\bar{x}, \bar{y}, \bar{z}, \bar{t}), \text{ etc.},$$

which give the relations connecting the space coordinates and the times used by two different observers. It is a simple matter to deduce from these equations the motion of the observer \bar{S} relative to S . \bar{S} 's position in space at any time is given in his own coordinates by $\bar{x} = \bar{y} = \bar{z} = 0$ and therefore in S 's coordinates we have

$$x = x(0, 0, 0, \bar{t}), \text{ etc.}$$

The elimination of \bar{t} gives (x, y, z) as functions of t , thus giving the motion of \bar{S} in S 's coordinates. Moreover every point which lies in the \bar{x} -coordinate surface of \bar{S} satisfies the equation $\bar{x} = 0$, and consequently the equation of this surface referred to S is

$$\bar{x}(x, y, z, t) = 0,$$

which is the equation of a surface moving relative to S . Similar results hold for the other coordinate surfaces of \bar{S} .

EXAMPLES.

1. Show that the motion of a particle in space is represented in space-time by a curve of the four-dimensional manifold. This curve is called the *world-line* of the particle.

2. Show that the motion of a surface in space is represented by a three-dimensional surface in space-time. This three-dimensional surface may be called the *history of the moving surface*.

3. If the space coordinates (x, y, z) of an observer S are orthogonal and Cartesian and those of another observer \bar{S} are connected with them by the formulæ

$$\begin{aligned}\bar{x} &= l(x - ut), & \bar{y} &= m(y - vt), & \bar{z} &= n(z - wt), \\ \bar{t} &= \alpha x + \beta y + \gamma z + \delta t,\end{aligned}$$

where l, u, α etc. are all constants, show that the observer \bar{S} is moving with a constant linear velocity relative to S and that his space-coordinate planes, as viewed by S , are moving parallel to themselves with constant velocities.

3. The principle of special relativity. The interval and the fundamental quadratic form.

Our next step is to introduce the hypotheses (or principles) of the special theory of relativity. For our purpose these may best be expressed in the following form, which we call *the principle of special relativity*:

The equations describing all physical phenomena in a definite reference system transform into equations of the same form when referred to any other system moving with a constant uniform velocity relative to the first.

Thus in the special theory of relativity there is a special class of privileged observers or reference systems, which are moving with constant velocities relative to each other. We call such special reference systems *Galilean systems of reference*.

We shall make two important deductions from this principle by means of two hypotheses which have been already accepted in classical mechanics and optics. Let S be one of our special class of reference frames and \bar{S} a second, and let us suppose that the space-coordinates in both are Cartesian.

(i) We first make the assumption that a particle, acted on by no forces, will describe a straight line in S 's frame of reference, that is, its space-coordinates referred to S will be linearly expressed in terms of the time. It results from our principle that the particle describes a straight line in \bar{S} 's frame and the new space-coordinates are also linearly expressed in terms of \bar{S} 's time. *Consequently the space-time coordinates of \bar{S} are linear functions of the space-time coordinates of S .*

(ii) Again we make the second assumption that the velocity of propagation of light referred to S is constant and independent of the motion of the light-source and we shall denote this

constant by c . Hence the velocity of propagation of light referred to \bar{S} is also a constant, which we shall denote by \bar{c} . We have therefore found from our principle the result that *the velocity of propagation of light referred to any one of our special reference frames is a constant.*

Let (x^1, x^2, x^3, t) be the Galilean coordinates of S and let us denote the corresponding coordinates of \bar{S} by barred letters. From our first assumption we see that the relations between the two sets of coordinates are

$$\bar{x}^r = \gamma_s^r x^s + \gamma_4^r t, \quad \bar{t} = \gamma_s^4 x^s + \gamma_4^4 t,$$

where the γ 's are constants. Moreover if a spherical light wave start from the point (x^1, x^2, x^3) at a time t , as observed by S , then at a time t' the wave front will lie on the surface

$$g_{mn} (x'^m - x^m) (x'^n - x^n) - c^2 (t' - t)^2 = 0, \quad \dots (5)$$

where g_{mn} is the fundamental metric tensor of S 's space. Also, as observed by \bar{S} , the wave front will lie on the surface

$$\bar{g}_{mn} (\bar{x}'^m - \bar{x}^m) (\bar{x}'^n - \bar{x}^n) - \bar{c}^2 (\bar{t}' - \bar{t})^2 = 0 \quad \dots (6)$$

where barred letters refer to \bar{S} 's observations. In other words, the equation (6) is a consequence of (5). Since the transformation from one set of coordinates to the other is linear, we therefore must have

$$\begin{aligned} & \bar{g}_{mn} (\bar{x}'^m - \bar{x}^m) (\bar{x}'^n - \bar{x}^n) - \bar{c}^2 (\bar{t}' - \bar{t})^2 \\ &= K [g_{mn} (x'^m - x^m) (x'^n - x^n) - c^2 (t' - t)^2], \end{aligned}$$

K being a constant. Now it can be immediately seen that by a change of \bar{S} 's units of space and time we can make not only $K = 1$ but also $\bar{c} = c$. This change of units we shall suppose to be always effected and we get

$$\begin{aligned} & \bar{g}_{mn} (\bar{x}'^m - \bar{x}^m) (\bar{x}'^n - \bar{x}^n) - c^2 (\bar{t}' - \bar{t})^2 \\ &= g_{mn} (x'^m - x^m) (x'^n - x^n) - c^2 (t' - t)^2. \end{aligned}$$

If the events (x^r, t) and (x'^r, t') are neighbouring events we may put

$$x'^r = x^r + dx^r, \quad t' = t + dt,$$

with corresponding changes in the barred letters, thus giving us the relation

$$\bar{g}_{mn} d\bar{x}^m d\bar{x}^n - c^2 d\bar{t}^2 = g_{mn} dx^m dx^n - c^2 dt^2,$$

or, as we may write it,

$$d\bar{\sigma}^2 - c^2 d\bar{t}^2 = d\sigma^2 - c^2 dt^2, \dots \dots \dots (7)$$

where $d\bar{\sigma}$, $d\sigma$ are the space elements of length for \bar{S} and S respectively. Also it is obvious that the relation (7) will still hold if we take any curvilinear space coordinates for S and \bar{S} instead of Cartesians.

We therefore see that the quadratic form

$$ds^2 = c^2 dt^2 - d\sigma^2 = c^2 dt^2 - g_{mn} dx^m dx^n \quad . \quad (8)$$

is an *invariant* for all changes of coordinates from one system S to another \bar{S} of our special class of observers. We call the invariant ds the *interval* between the two events (x^r, t) and $(x^r + dx^r, t + dt)$. If we use generalised coordinates as described in the previous section we see that ds^2 will be a homogeneous quadratic form in dx^α , that is,

$$ds^2 = a_{\alpha\beta} dx^\alpha dx^\beta, \dots \dots \dots (9)$$

where $a_{\alpha\beta}$ is a function of the coordinates symmetric in α, β . Since ds^2 is an invariant we deduce that $a_{\alpha\beta}$ is a double tensor, which we call the *fundamental tensor*, and we call (9) the *fundamental quadratic form*. If we use one of our special frames of reference we know that our quadratic form takes the particular form (8). Moreover if we take a Galilean reference system in which the space-coordinates (x^1, x^2, x^3) are orthogonal and Cartesian, the fundamental quadratic form becomes

$$ds^2 = c^2 dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \dots \dots (10)$$

We have now deduced a fundamental quadratic form in the four-dimensional domain similar to that found for the distance between two points in three-dimensional space. Consequently we can write down the Christoffel symbols for such a form; they are defined by the equations

$$[\alpha\beta, \gamma] = \frac{1}{2} \left(\frac{\partial a_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial a_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial a_{\alpha\beta}}{\partial x^\gamma} \right), \dots \dots (11)$$

$$\gamma \quad \searrow \quad \gamma\delta$$

$$a_{\gamma\delta} [\alpha\beta, \delta].$$

$a^{\gamma\delta}$ being the cofactor of $a_{\gamma\delta}$ in the determinant $|a_{\gamma\delta}|$ divided by $|a_{\gamma\delta}|$. In exactly the same manner as in Part III we can

define the intrinsic and covariant derivatives of a tensor-field by means of these Christoffel symbols, and these derivatives will have similar properties in four dimensions to those in three. For example, we define the intrinsic derivative of A^α_β with respect to a parameter u (cf. pp. 146, 147) as

$$\frac{\delta A^\alpha_\beta}{\delta u} = \frac{dA^\alpha_\beta}{du} + \left\{ \begin{matrix} \alpha \\ \mu \nu \end{matrix} \right\} A^\mu_\beta \frac{dx^\nu}{du} - \left\{ \begin{matrix} \mu \\ \beta \nu \end{matrix} \right\} A^\alpha_\mu \frac{dx^\nu}{du} \quad (12)$$

and its covariant derivative with respect to x^γ as

$$A^\alpha_{\beta,\gamma} = \frac{\partial A^\alpha_\beta}{\partial x^\gamma} + \left\{ \begin{matrix} \alpha \\ \mu \gamma \end{matrix} \right\} A^\mu_\beta - \left\{ \begin{matrix} \mu \\ \beta \gamma \end{matrix} \right\} A^\alpha_\mu. \quad (13)$$

EXAMPLES.

1. Show that in a Galilean system of reference with Cartesian space axes the Christoffel symbols are zero, and deduce that the intrinsic and covariant derivatives reduce in such coordinates to ordinary derivatives.

2. If a particle moves from one event A to another B with a velocity v relative to an observer S , show that the interval between the two events is given by

$$s = \int_A^B \sqrt{c^2 - v^2} dt.$$

3. If a particle moves from one event to another with uniform velocity, show that the interval s between the two events, divided by c , equals the time as measured by a clock carried with the particle. Thus s is sometimes called the *local time*.

[Choose a Galilean coordinate system whose origin always coincides with the particle and use Ex. 2.]

4. *Lorentz transformation.* If (x^1, x^2, x^3) are orthogonal Cartesian space-coordinates and t the time of a Galilean system S , and the coordinates of another system \bar{S} are connected with those of S by the formulae

$$\bar{x}^1 = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} (x^1 - ut), \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = x^3, \\ 1 - \frac{u^2}{c^2} \quad t -$$

show that \bar{S} is moving along the x^1 -axis of S with uniform velocity u and that $\bar{x}^1, \bar{x}^2, \bar{x}^3$ are orthogonal Cartesian space axes for \bar{S} . This transformation therefore gives the relation between the space and time co-ordinates of two special systems of reference.

[Show that $ds^2 = c^2 d\bar{t}^2 - (d\bar{x}^1)^2 - (d\bar{x}^2)^2 - (d\bar{x}^3)^2$.]

5. Show that in a Galilean system of reference

$$\begin{aligned} a_{rs} &= -g_{rs}, & a_{r4} &= 0, & a_{44} &= c^2, & a &= -c^2 g \\ a^{rs} &= -g^{rs}, & a^{r4} &= 0, & a^{44} &= \frac{1}{c^2}, \end{aligned}$$

where the g 's refer to the space coordinates and

$$a_{\alpha\beta} |, \quad g = |g_{rs}|.$$

[Note that a is negative.]

4. Local coordinate systems and their transformations.

If we consider a point P of space which is moving in any way we can associate with this point a set of special coordinate systems which we are now going to describe. If P is moving with a velocity v at the instant under consideration we can take a Galilean frame of reference which has the same velocity v at this instant so that the point P is for the moment at rest in our reference system and we shall determine any event by means of the space and time coordinates (x^r, t) of our frame. Such a system of coordinates we shall call *local coordinates for the point P* . We see that this does not determine our local coordinates uniquely and we shall find what is the transformation from one local coordinate system (x^r, t) to another (\bar{x}^r, \bar{t}) .

Since the two coordinate systems are moving with the same uniform velocity we see that their times are identical, namely the time measured by a clock carried with them. Thus $\bar{t} = t$. Also the transformation of space coordinates will not involve the time as the space frames are not moving relative to each other. Hence $\bar{x}^r = \bar{x}^r(x^1, x^2, x^3)$ and we have the result that the transformation from one local system to another is of the form

$$\bar{x}^r = \bar{x}^r(x^1, x^2, x^3), \quad \bar{t} = t. \quad (14)$$

If we put $x^4 = t$, $\bar{x}^4 = \bar{t}$ for the present, we see that (14) may be written in the equivalent form

$$\bar{x}^r = \bar{x}^r(x^1, x^2, x^3), \quad \bar{x}^4 = x^4. \quad (15)$$

Let us investigate how the components of world vectors and world tensors transform when we change from one system of local coordinates to another at the same point. We consider first a contravariant world vector A^α . From the law of transformation we have

$$\bar{A}^\alpha = A^\mu \frac{\partial \bar{x}^\alpha}{\partial x^\mu},$$

from which we see that the transformation (15) gives

$$A^r = A^m \frac{\partial \bar{x}^r}{\partial x^m} \quad A^4 = A^4. \quad (16)$$

In other words, under a local coordinate transformation (A^1, A^2, A^3) transforms as a space vector whilst A^4 transforms as an invariant. We can therefore conclude from this that we can always define a world vector A^α such that in any local coordinate system its first three components are those of a given space vector and its fourth component is a given space invariant. We shall have of course similar results for the covariant vector A_α .

We next consider the double world tensor $A^{\alpha\beta}$. Its law of transformation is

$$\bar{A}^{\alpha\beta} = A^{\mu\nu} \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \frac{\partial \bar{x}^\beta}{\partial x^\nu}.$$

When the transformation is from one local system to another we see from (15) that

$$\begin{aligned} \bar{A}^{rs} &= A^{mn} \frac{\partial \bar{x}^r}{\partial x^m} \frac{\partial \bar{x}^s}{\partial x^n}, \quad \bar{A}^{r4} = A^{m4} \frac{\partial \bar{x}^r}{\partial x^m}, \\ A^{1r} &= A^{4r}, \quad \frac{\partial \bar{x}^r}{\partial x^m}, \quad \bar{A}^{44} = A^{44}. \end{aligned}$$

We can therefore divide up the components of $A^{\alpha\beta}$ into groups

$$A^{\alpha\beta} \quad \left\{ \begin{array}{l} A^{rs}, \quad A^{r4} \\ A^{4s}, \quad A^{44} \end{array} \right\},$$

of which A^{rs} transforms as a double space tensor, A^{r4} and A^{4r} as contravariant space vectors and A^{44} as a space invariant under local coordinate transformations. We conclude that we can always define a double world tensor $A^{\alpha\beta}$ such that in any local coordinate system the components A^{rs} form a given space tensor, A^{r4} , A^{4r} form given contravariant space vectors and A^{44} is a given space invariant, and we see that the particular local system used for the definition is immaterial. Similar results hold for the double tensor $A_{\alpha\beta}$.

EXAMPLES.

1. Show that in a local system

$$a_{44} = c^2, \quad a_{4r} = 0, \quad a^{44} = \frac{1}{c^2}, \quad a^{4r} = 0, \quad \frac{\partial a_{mn}}{\partial t} = 0.$$

2. Deduce from Ex. 1 that the Christoffel symbols $[\alpha\beta, \gamma]$ and $\left. \begin{array}{l} \gamma \\ \alpha\beta \end{array} \right\}$ both vanish when one of the α, β, γ equals 4.

3. Deduce from Ex. 2 that if we calculate the vector $A^{\alpha\beta}_{\cdot, \beta}$ in a local coordinate system then its first three components are

$$A^{rs}_{\cdot, s} + \frac{\partial A^{r4}}{\partial t}$$

and its fourth component is

$$\left(A^{4m}_{\cdot, m} + \frac{\partial A^{44}}{\partial t} \right).$$

Here $A^{r,s}_{\dots}$ and $A^{1,m}_{\dots}$ refer to the space covariant derivatives of A^{rs} and A^{4r} , which we have seen to be a space tensor and a space vector relative to local transformations.

$$\begin{aligned} \text{We have } A^{\alpha\beta}_{\dots,\beta} &= \frac{\partial A^{\alpha\beta}}{\partial x^\beta} + \left\{ \begin{array}{c} \alpha \\ \mu\beta \end{array} \right\} A^{\mu\beta} + \left\{ \begin{array}{c} \beta \\ \mu\beta \end{array} \right\} A^{\alpha\mu} \\ &- \frac{\partial A^{\alpha m}}{\partial x^m} - \frac{\partial A^{\alpha 4}}{\partial t} + \left\{ \begin{array}{c} \alpha \\ mn \end{array} \right\} A^{mn} + \left\{ \begin{array}{c} m \\ nm \end{array} \right\} A^{\alpha n}, \text{ by using Ex. 2.} \end{aligned}$$

4. Show that for the point P at the instant under consideration

$$\frac{ds}{dt} = c, \quad \frac{d^2s}{dt^2} = 0,$$

when a local coordinate system is used.

[If v is the velocity of a point in a Galilean frame we see from (8) that $\frac{ds}{dt} = \sqrt{c^2 - v^2}$ and so $\frac{d^2s}{dt^2} = -v \frac{dv}{dt} / \sqrt{c^2 - v^2}$. Now in a local coordinate system for the point $v = 0$.]

5. Relativistic dynamics of a particle.

Let us suppose that a particle is moving under the action of a certain force system. The representation of the motion in space-time will be a curve, which is called the *world-line of the particle*, and along this curve we can take as parameter the interval, which as before we shall denote by s .

Let us consider the vector

$$\alpha^\mu = \frac{dx^\mu}{ds}. \quad \dots \dots \dots (17)$$

Its components referred to a Galilean system are

$$\left(\frac{dx^1}{ds}, \quad \frac{dx^2}{ds}, \quad \frac{dx^3}{ds}, \quad \frac{dt}{ds} \right).$$

Now if we take a local coordinate system for the particle we easily see that $\frac{ds}{dt} = c$ in such a system, and the components of α^μ become

$$\left(0, 0, 0, \frac{1}{c} \right). \quad \dots \dots \dots (18)$$

We shall call the vector (17) the *world velocity vector*.

Again, let us consider the vector

$$\gamma^\mu = \frac{d\alpha^\mu}{ds} = \frac{d^2x^\mu}{ds^2} + \left\{ \begin{matrix} \mu \\ \rho \sigma \end{matrix} \right\} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds}, \quad \dots \quad (19)$$

which is the intrinsic derivative of α^μ with respect to s . To find its components in a local coordinate system we use the fact that in this system $\frac{d^2s}{dt^2} = 0$ (Ex. 4 above). Hence for the instant under consideration the components of γ^μ referred to a local reference frame are

$$\left(\frac{1}{c^2} \frac{d^2x^r}{dt^2}, 0 \right). \quad \dots \quad (20)$$

We shall call γ^μ the *world acceleration vector*.

We shall investigate the equations of motion of a particle under a given force system. We take a local frame of reference for the particle, that is, the observer is moving for the instant with the same velocity as the particle and we shall assume that the classical laws of dynamics hold in this special coordinate system. Consequently we have

$$m_0 \frac{d^2x^r}{dt^2} = X^r, \quad \dots \quad (21)$$

where m_0 is an invariant called the *proper* or *local mass* of the particle and X^r is the force vector in the local system. Let us introduce a world vector F^μ whose components in the local system are

$$\left(\frac{1}{c^2} X^r, 0 \right). \quad \dots \quad (22)$$

We have seen that this is possible in the previous section. By reference to (20), (21) and (22) we see that we have the tensor equation

$$m_0 \gamma^\mu = F^\mu. \quad \dots \quad (23)$$

For this vector equation is seen to be true in the local reference frame and is therefore true in all coordinate systems. We may call F^μ the *world force vector* and (23) gives the relativistic equations of motion.

EXAMPLES.

1. Show that $a_{\mu r} \alpha^\mu \alpha^r = 1$.

[From analogy with three dimensions we say that α^μ is a *unit* world vector.]

2. Deduce that in a Galilean system $c^2 (\alpha^4)^2 = 1 + g_{mn} \alpha^m \alpha^n$.

3. Show that $a_{\mu r} \gamma^\mu \alpha^r = 0$.

[This relation is an invariant relation which is true in the local reference frame. From analogy with three dimensions we say that γ^μ is orthogonal to α^μ .]

4. Show that $a_{\mu r} F^\mu \alpha^r = 0$ and deduce that in a Galilean system

$$c^2 F^4 = g_{mn} F^m \frac{dx^n}{dt}.$$

5. Show that in a local reference frame the associated vector $\alpha_\mu = a_{\mu r} \alpha^r$ has the values $(0, 0, 0, c)$.

6. Dynamics of a continuous medium.

We shall now find the relativistic form of the equations of motion of a continuous medium. Let us consider a definite point of the medium and let us take a local coordinate system for this point. We assume, as before, that the classical equations of motion hold for this special reference frame. The stress in the medium is defined by a space-tensor E^{rs} and the body force by a space-vector X^r . The equations of motion in the local system are

$$E^{r,s}_{\cdot\cdot,s} + \varrho_0 (X^r - f^r) = 0 \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where ϱ_0 is the invariant *proper density* and f^r is the space acceleration vector. In addition we have the equation of continuity

$$(\varrho_0 u^r)_{,r} + \frac{\partial \varrho_0}{\partial t} = 0 \quad . \quad . \quad . \quad . \quad . \quad (25)$$

u^r being the velocity vector. Our object is to express these equations in terms of *world tensors*.

For this purpose we shall introduce the vectors α^μ , γ^μ and F^μ of the previous section and let us introduce a new world tensor $E^{\mu\nu}$ defined by the conditions that its components in the local reference system are

$$E^{\mu\nu} = \frac{1}{c^2} E^{rs}, \quad 0 \quad , \quad 0 \quad (26)$$

That this is possible we have seen in § 4 (p. 292). In terms of these world tensors we easily see that the equations (24) and (25) may be written

$$\left. \begin{aligned} E^{\mu\nu}_{\dots,\nu} + \varrho_0 (F^\mu - \gamma^\mu) &= 0, \\ (\varrho_0 \alpha^\mu)_{,\mu} &= 0, \end{aligned} \right\} \dots \dots (27)$$

which are both tensor equations in space-time and therefore true in all coordinate systems.

These two equations can be combined into a single equation as follows. We have

$$\gamma^\mu = \frac{\delta \alpha^\mu}{\delta s} = \alpha^\mu_{\dots,\nu} \alpha^\nu.$$

Hence, making use of the second equation of (27),

$$\varrho_0 \gamma^\mu = \varrho_0 \alpha^\mu_{\dots,\nu} \alpha^\nu = (\varrho_0 \alpha^\mu \alpha^\nu)_{,\nu} - \alpha^\mu (\varrho_0 \alpha^\nu)_{,\nu} = (\varrho_0 \alpha^\mu \alpha^\nu)_{,\nu}.$$

Thus if we put

$$T^{\mu\nu} = E^{\mu\nu} - \varrho_0 \alpha^\mu \alpha^\nu \dots \dots \dots (28)$$

we see that we have the tensor equation

$$T^{\mu\nu}_{\dots,\nu} + \varrho_0 F^\mu = 0. \dots \dots \dots (29)$$

This equation is equivalent to the equations (27). For, if we multiply (29) by α_μ and sum with respect to μ from 1 to 4, we find that

$$T^{\mu\nu}_{\dots,\nu} \alpha_\mu = -\varrho_0 F^\mu \alpha_\mu = 0$$

and consequently

$$(\varrho_0 \alpha^\nu)_{,\nu} = (\varrho_0 \alpha^\mu \alpha^\nu)_{,\nu} \alpha_\mu = -T^{\mu\nu}_{\dots,\nu} \alpha_\mu = 0,$$

that is, the equation of continuity is satisfied. We can now prove that the first of (27) is a consequence of (29) by merely reversing the steps.

The tensor equation (29) is therefore the final relativistic form of the equations of motion of a continuous medium.

EXAMPLES.

1. Show that $E^{\mu\nu} \alpha_\nu = 0$ and that $E^{\mu\nu}_{\dots,\nu} \alpha_\mu = 0$.
2. Show that $E^{\mu\nu} \alpha_{\mu,\nu} = 0$.
3. Show that the tensor equation (29) can be written in the form

$$E^{\mu\nu}_{\dots,\nu} + \varrho_0 (F^\mu - \gamma^\mu) = (\varrho_0 \alpha^\nu)_{,\nu} \alpha^\mu.$$

7. The electromagnetic equations.

In order to express the electrodynamical equations in terms of world tensors we must introduce the fundamental skew-symmetric world tensors of the fourth order. We denote them by $\varepsilon_{\lambda\mu\nu\varrho}$ and $\varepsilon^{\lambda\mu\nu\varrho}$ and they have properties analogous to the corresponding ε -systems in three dimensions. They are defined as follows:

(i) $\varepsilon_{\lambda\mu\nu\varrho}$ vanishes if any two indices are equal and equals $\pm \sqrt{-a}$ according as $\lambda, \mu, \nu, \varrho$ is an even or odd permutation of 1, 2, 3, 4.

(ii) $\varepsilon^{\lambda\mu\nu\varrho}$ vanishes if any two indices are equal and equals $\pm 1/\sqrt{-a}$ according as $\lambda, \mu, \nu, \varrho$ is an even or odd permutation of 1, 2, 3, 4. We choose $\sqrt{-a}$ instead of \sqrt{a} since a is always negative in the formulae of the theory of relativity.

We prove in the same way as for three dimensions that they are world tensors.

If we have a dielectric in motion we may take a special local reference frame with respect to a given point of the dielectric. This point is *at rest* for the instant relative to our reference system and we suppose that the classical electromagnetic equations hold in such a system. The classical electromagnetic field is defined by the space vectors D^r, E_r, B^r, H_r and the equations in the local system are

$$\left. \begin{aligned} \varepsilon^{rst} E_{s,t} - \frac{1}{c} \frac{\partial B^r}{\partial t} &= 0, \\ \frac{1}{c} B^r_{,r} &= 0, \end{aligned} \right\} \dots \dots (30)$$

and

$$\left. \begin{aligned} \varepsilon^{rst} H_{s,t} + \frac{1}{c} \frac{\partial D^r}{\partial t} + \frac{1}{c} C^r &= 0, \\ -\frac{1}{c} D^r_{,r} + \frac{1}{c} \varrho_0 &= 0, \end{aligned} \right\} \dots \dots (31)$$

where C^r is the vector giving the conduction current. To these equations we must add the relations

$$D^r = \varepsilon^{rs} E_s, \quad B^r = \mu^{rs} H_s \quad \dots \dots (32)$$

connecting these four vectors. We shall now express these equations in terms of world-tensors.

Let us introduce the two skew-symmetric world tensors $F^{\mu\nu}$ and $G^{\mu\nu}$ whose components in the local system are

$$\begin{aligned} F^{\mu\nu} = \begin{pmatrix} \varepsilon^{rst} E_t, & \frac{1}{c} B^r \\ -\frac{1}{c} B^s, & 0 \end{pmatrix} \quad G^{\mu\nu} = \begin{pmatrix} H_t, & -\frac{1}{c} D^r \\ \frac{1}{c} D^s, & \end{pmatrix} \end{aligned} \quad (33)$$

and also the world vector S^μ whose local components are

$$S^\mu = \left(\frac{1}{c} C^1, \frac{1}{c} C^2, \frac{1}{c} C^3, \frac{1}{c} C^0 \right) \dots \dots \dots (34)$$

We easily see that the equations (30) and (31) take the tensor form

$$F^{\mu\nu}_{;\nu} = 0, \quad G^{\mu\nu}_{;\nu} = S^\mu, \dots \dots \dots (35)$$

and these are the relativistic forms of the electromagnetic equations.

We have to deduce the relations between the two tensors $F^{\mu\nu}$, $G^{\mu\nu}$ which result from the equations (32). These relations are

$$cF^{14} = \sqrt{g}(\mu^{11}G^{23} + \mu^{12}G^{31} + \mu^{13}G^{12}), \text{ etc.}$$

and

$$cG^{14} = -\sqrt{g}(\varepsilon^{11}F^{23} + \varepsilon^{12}F^{31} + \varepsilon^{13}F^{12}), \text{ etc.}$$

If we put $\mu^{\varrho\sigma}$ and $\varepsilon^{\varrho\sigma}$ for the world tensors whose components in the local system are

$$\mu^{\varrho\sigma} = \left\{ \begin{matrix} \mu^{rs}, & 0 \\ 0, & 0 \end{matrix} \right\}, \quad \varepsilon^{\varrho\sigma} = \left\{ \begin{matrix} \varepsilon^{rs}, & 0 \\ 0, & 0 \end{matrix} \right\}, \dots \dots \dots (36)$$

and if we remember that the special components of α^ϱ and α_ϱ are $\left(0, 0, 0, \frac{1}{c}\right)$ and $(0, 0, 0, c)$ respectively, the relations take the tensor forms

$$\left. \begin{aligned} 2F^{\varrho\sigma}\alpha_\sigma &= \mu^{\varrho\lambda}\varepsilon_{\lambda\mu\nu\sigma}G^{\mu\nu}\alpha^\sigma, \\ 2G^{\varrho\sigma}\alpha_\sigma &= -\varepsilon^{\varrho\lambda}\varepsilon_{\lambda\mu\nu\sigma}F^{\mu\nu}\alpha^\sigma. \end{aligned} \right\} \dots \dots \dots (37)$$

EXAMPLES.

1. If we define E_ϱ , D^ϱ , H_ϱ , B^ϱ as the world tensors whose local components are $(E_r, 0)$, $(D^r, 0)$, $(H_r, 0)$, $(B^r, 0)$ respectively, show that

$$\begin{aligned} F^{\lambda\mu} &= \varepsilon^{\lambda\mu\nu\varrho}E_\nu\alpha_\varrho + B^\lambda\alpha^\mu - B^\mu\alpha^\lambda, \\ G^{\lambda\mu} &= \varepsilon^{\lambda\mu\nu\varrho}H_\nu\alpha_\varrho - D^\lambda\alpha^\mu + D^\mu\alpha^\lambda. \end{aligned}$$

2. If C^e is the world vector whose local components are $(\frac{1}{c} C^1, \frac{1}{c} C^2, \frac{1}{c} C^3, 0)$ show that $S^e = C^e + c_0 \alpha^e$.

3. If we define the tensor $H_{\mu\nu}$ by means of the formula

$$H_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma},$$

show that the first of the equations (35) may be written in the equivalent form

$$\varepsilon^{\lambda\mu\nu\rho} H_{\mu\nu, \rho} = 0.$$

4. Prove the relation

$$G^{\mu\nu} \alpha_\nu = - \varepsilon^{\mu\lambda} H_{\lambda\nu} \alpha^\nu.$$

5. Show that the local components of $H_{\mu\nu}$ are

$$\begin{pmatrix} \varepsilon_{rst} B^t, & cE_r \\ cE_s, & 0 \end{pmatrix}$$

6. If we take the associated tensor $G_{\mu\nu} = a_{\mu\alpha} a_{\nu\beta} G^{\alpha\beta}$, show that its local components are

$$\begin{pmatrix} \varepsilon_{rst} H^t, & cD_r \\ -cD_s, & 0 \end{pmatrix}$$

EXAMPLES XXI.

1. An observer \bar{S} is moving with uniform velocity u relative to another observer S . If S uses orthogonal Cartesian space axes and \bar{S} uses as space coordinate planes the planes which coincide with S 's coordinate planes at the time $t = \bar{t} = 0$, show that the transformation between the two coordinate systems is

$$\begin{aligned} \bar{x}^1 &= \left(1 + \frac{u_1^2}{c^2 - u^2}\right)^{\frac{1}{2}} (x^1 - u_1 t), & \bar{x}^2 &= \left(1 + \frac{u_2^2}{c^2 - u^2}\right)^{\frac{1}{2}} (x^2 - u_2 t), \\ \bar{x}^3 &= \left(1 + \frac{u_3^2}{c^2 - u^2}\right)^{\frac{1}{2}} (x^3 - u_3 t), & \bar{t} &= \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} \left(t - \frac{u_r x^r}{c^2}\right); \end{aligned}$$

where u_1, u_2, u_3 are the components of u relative to S 's coordinates. Also show that the cosines of the angles between \bar{S} 's space coordinate planes as viewed by \bar{S} are

$$\frac{u_2 u_3}{c^2 - u^2} \left(1 + \frac{u_3^2}{c^2 - u^2}\right)^{-\frac{1}{2}} \left(1 + \frac{u_1^2}{c^2 - u^2}\right)^{-\frac{1}{2}}, \text{ etc.}$$

[We must assume $\bar{x}^1 = e_1 (x^1 - u_1 t)$, etc. $\bar{t} = \alpha_r x^r + \beta t$. Use fact that $ds^2 = c^2 dt^2 - \sum (dx^i)^2 = c^2 d\bar{t}^2 - \sum (d\bar{x}^i)^2 - 2 \sum \bar{a}_i d\bar{x}^i d\bar{x}^1$ for all values of the differentials, $\bar{a}_1, \bar{a}_2, \bar{a}_3$ being the cosines of the angles in question. Also $u^2 = u_1^2 + u_2^2 + u_3^2$.]

2. *Lorentz transformations.* If \bar{S} is moving along S 's x^1 -axis show that the transformation of Ex. 1 becomes

$$\bar{x}^1 = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} (x^1 - ut), \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = x^3,$$

$$\bar{t} = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} \left(t - \frac{ux^1}{c^2}\right),$$

and that \bar{S} 's space coordinates are also orthogonal.

3. Show that the equations in generalised coordinates of the path of a particle under no forces is

$$\frac{d^2 x^\alpha}{ds^2} + \left\{ \begin{array}{c} \alpha \\ \mu \nu \end{array} \right\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0.$$

[These are called the *geodesics* of space-time from the similarity of their equations to those of the geodesics of a surface.]

4. In a Galilean reference frame show that the equations of motion of a particle may be written

$$\frac{d}{dt} \left(\frac{m_0}{\sqrt{c^2 - v^2}} \frac{dx^r}{dt} \right) = \sqrt{c^2 - v^2} F^r,$$

$$\frac{d}{dt} \left(\frac{m_0}{\sqrt{c^2 - v^2}} \right) = \sqrt{c^2 - v^2} F^4 \quad \sqrt{c^2 - v^2} g_{mn} F^m \frac{dx^n}{dt}$$

where v is the velocity of the particle relative to the reference system.

[Use the fact that in a Galilean system $\frac{ds}{dt} = \sqrt{c^2 - v^2}$.]

5. Deduce from Ex. 4 that the equations of motion of a particle in a Galilean frame of reference are

$$\frac{d}{dt} \left(m \frac{dx^r}{dt} \right) = X^r, \quad \frac{d}{dt} (m) = \frac{1}{c^2} g_{mn} X^n \frac{dx^n}{dt}$$

where

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \quad X^r = c^2 F^r \sqrt{1 - v^2/c^2}.$$

[The student will notice the similarity of these equations to those of classical dynamics. We may interpret m as the mass and X^r as the force vector relative to the reference system, and we see that the rate of change of mass is proportional to the rate at which the force is doing work.]

6. If a particle is acted on by no forces, show that the quantity $\frac{m_0}{\sqrt{1 - v^2/c^2}}$ is constant as viewed by an observer S of our special class and show also that its velocity relative to this observer is constant.

7. Show that the relativistic equations of a perfect fluid are

$$T^{\mu\nu}_{;\nu} + \varrho_0 F^\mu = 0$$

where

$$T^{\mu\nu} = p a^{\mu\nu} - (p + \varrho_0) \alpha^\mu \alpha^\nu.$$

The invariant p is the local pressure.

[Show that the stress tensor is $E^{\mu\nu} = p(a^{\mu\nu} - \alpha^\mu \alpha^\nu)$.]

8. In the motion of a continuous medium show that

$$T^{\mu\nu} \alpha_\mu \alpha_\nu = -\varrho_0, \quad -\varrho_0 \alpha^\mu = T^{\mu\nu} \alpha_\nu, \quad a_{\mu\nu} T^{\mu\nu} = a_{\mu\nu} E^{\mu\nu} - \varrho_0.$$

9. In the electromagnetic field show that we have the relations

$$\varepsilon^{\lambda\mu\nu\sigma} (a_{\sigma\lambda} H_{\mu\nu} - \mu_{\sigma\lambda} G_{\mu\nu}) \alpha_\sigma = 0,$$

connecting the tensors $H_{\mu\nu}$ and $G_{\mu\nu}$.

10. In the electromagnetic equations if the dielectric is isotropic show that we have the relations

$$(G_{\mu\nu} - \varepsilon H_{\mu\nu}) \alpha^\nu = 0, \quad \varepsilon^{\lambda\mu\nu\varrho} (H_{\mu\nu} - \mu G_{\mu\nu}) \alpha_\varrho = 0,$$

ε and μ being invariants.

11. Prove that the current vector S^μ satisfies the equation $S^\mu_{;\mu} = 0$.

12. If we define a world vector φ^μ satisfying the relations

$$G_{\mu\nu} = \varphi_{\mu,\nu} - \varphi_{\nu,\mu}, \quad \varphi^\mu_{;\mu} = 0,$$

show that φ^μ satisfies the equation

$$a^{\sigma\tau} \varphi^\mu_{;\sigma\tau} = S^\mu.$$

The vector φ^μ is called the electromagnetic potential world vector.

13. If we define the world tensor $\varkappa^{\mu\nu}$ as that whose components in the local system are

$$\varkappa^{\mu\nu} = \begin{Bmatrix} \varkappa^{rs}/c, & 0 \\ 0 & , & 0 \end{Bmatrix},$$

\varkappa^{rs} being the local conduction coefficients, show that S^μ is given by

$$S^\mu = \varkappa^{\mu\nu} E_\nu + \varrho_0 \alpha^\mu.$$

14. Show that the formula of Ex. 13 can be written in the equivalent forms

$$S^\mu = \frac{1}{2} \varkappa^{\mu\nu} \varepsilon_{\nu\varrho\sigma\tau} F^{\varrho\sigma} \alpha^\tau + \varrho_0 \alpha^\mu, \quad S^\mu = \varkappa^{\mu\nu} H_{\nu\varrho} \alpha^\varrho + \varrho_0 \alpha^\mu.$$

15. For the relativistic form of the equations of the *electron theory* we have merely to remark that in the local system D_r is identical with E_r and B_r with H_r . Show that for the electron theory $G_{\mu\nu} = H_{\mu\nu}$ and $S^\mu = \varrho_0 \alpha^\mu$ and deduce that the equations are

$$\varepsilon^{\lambda\mu\nu\varrho} H_{\mu\nu,\varrho} = 0, \quad a^{\mu\nu} H_{\lambda\mu,\nu} = \varrho_0 \alpha_\lambda.$$

16. If we transform from the local frame of reference x^μ to another Galilean system \bar{x}^μ in accordance with the Lorentz transformation

$$\bar{x}^1 = k(x^1 - vx^4), \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = x^3, \quad x^4 = k(x^4 - vx^1/c^2),$$

$$k = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c},$$

show that the new values for \bar{E}^r and \bar{H}^r in the electron theory are

$$\bar{E}^1 = E^1, \quad \bar{E}^2 = k(E^2 - \beta H^3), \quad \bar{E}^3 = k(E^3 + \beta H^2),$$

$$\bar{H}^1 = H^1, \quad \bar{H}^2 = k(H^2 + \beta E^3), \quad \bar{H}^3 = k(H^3 - \beta E^2).$$

[Use the fact that $H_{\mu\nu}$ is a double tensor.]

APPENDIX

Orthogonal Curvilinear Coordinates in Mathematical Physics

1. The classical notation.

In the text we have dealt with the use of curvilinear coordinates in mathematical physics by the methods of the tensor calculus. Since orthogonal curvilinear coordinates are introduced into the ordinary text-books on mathematical physics without using tensor methods or tensor notation, it will be useful to translate our results into the classical notation employed in such books*.

In this notation we take three families of surfaces

$$f_1(x, y, z) = \alpha, \quad f_2(x, y, z) = \beta, \quad f_3(x, y, z) = \gamma, \quad . \quad (1)$$

which cut mutually at right angles, x, y, z being Cartesian coordinates. A point is determined by the three coordinates α, β, γ and the line-element is given by the formula

$$ds^2 = \frac{d\alpha^2}{h_1^2} + \frac{d\beta^2}{h_2^2} + \frac{d\gamma^2}{h_3^2}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

h_1, h_2, h_3 being functions of α, β, γ . That is, connecting the notation with that of the text,

$$\left. \begin{aligned} x^1 &= \alpha, \quad x^2 = \beta, \quad x^3 = \gamma, \\ g_{11} &= \frac{1}{h_1^2}, \quad g_{22} = \frac{1}{h_2^2}, \quad g_{33} = \frac{1}{h_3^2}, \quad g_{mn} = 0 \quad (m \neq n), \\ g &= \frac{1}{h_1^2 h_2^2 h_3^2}. \end{aligned} \right\} . \quad (3)$$

* The notation we use here is that employed in books like Love's *Mathematical Theory of Elasticity*.

Also the associated tensor g^{mn} has the values

$$g^{11} = h_1^2, \quad g^{22} = h_2^2, \quad g^{33} = h_3^2, \quad g^{mn} = 0 \quad (m \neq n). \quad (4)$$

We easily verify that the Christoffel symbols are

$$\begin{aligned} \frac{k}{ij} = 0, \quad \frac{j}{ii} &= \frac{h_i^2}{h_i^3} \frac{\partial h_i}{\partial x^j}, \quad \left\{ \frac{i}{ij} \right\} = -\frac{1}{h_i} \frac{\partial h_i}{\partial x^j}, \\ \frac{i}{ii} &= -\frac{1}{h_i} \frac{\partial h_i}{\partial x^i}, \end{aligned} \quad (5)$$

where i, j, k are unequal and the summation convention has been dropped (cf. Ex. 9, p. 155).

2. The physical components of vectors and tensors.

Let us denote by $\varepsilon_{(i)}^r$ the unit vector tangent to the i^{th} coordinate curve. We see immediately that

$$\varepsilon_{(1)}^r = h_1 \delta_1^r, \quad \varepsilon_{(2)}^r = h_2 \delta_2^r, \quad \varepsilon_{(3)}^r = h_3 \delta_3^r. \quad \dots \quad (6)$$

If we are given a vector A^r , instead of taking A^1, A^2, A^3 as the components of the vector, in the classical notation we take as the components the projections of the vector on the tangents to the coordinate curves. We shall denote these components by $A_\alpha, A_\beta, A_\gamma$ and we have

$$A_\alpha = A_r \varepsilon_{(1)}^r, \quad A_\beta = A_r \varepsilon_{(2)}^r, \quad A_\gamma = A_r \varepsilon_{(3)}^r,$$

where A_r equals $g_{rs} A^s$ and is the associated vector of A^r . Thus

$$\left. \begin{aligned} A_\alpha &= h_1 A_1 = \frac{A^1}{h_1}, & &= h_2 A_2 = \frac{A^2}{h_2}, \\ A_\gamma &= h_3 A_3 & &= \frac{A^3}{h_3}. \end{aligned} \right\} \quad \dots \quad (7)$$

We call these components *the physical components* of the vector to distinguish them from the tensor components. Since $A_\alpha, A_\beta, A_\gamma$ are the projections of A^r on the tangents to the coordinate curves, we see that the physical components of a vector are simply its components referred to a Cartesian system of axes coinciding with the tangents mentioned.

Similarly, if we are given a double tensor A^{rs} , we define its *physical components* as $A_{\alpha\alpha}, A_{\beta\gamma}$, etc., where

$$A_{\alpha\alpha} = A_{rs} \varepsilon_{(1)}^r \varepsilon_{(1)}^s, \quad A_{\beta\gamma} = A_{rs} \varepsilon_{(2)}^r \varepsilon_{(3)}^s, \quad \text{etc.},$$

A_{rs} being the associated tensor of A^{rs} . Hence

$$A_{\alpha\alpha} = h_1^2 A_{11} = \frac{A^{11}}{h_1^2}, \quad A_{\beta\gamma} = h_2 h_3 A_{23} = \frac{A^{23}}{h_2 h_3} \quad \text{etc.} \quad (8)$$

An interpretation of the physical components of a double tensor can be given similar to that given in the case of a vector. The expressions

$$A_{\alpha\alpha} = A_{rs} \varepsilon_{(1)}^r \varepsilon_{(1)}^s, \quad A_{\beta\gamma} = A_{rs} \varepsilon_{(2)}^r \varepsilon_{(3)}^s, \quad \text{etc.}$$

are obviously *invariants* for all transformations of axes so long as $\varepsilon_{(1)}^r, \varepsilon_{(2)}^r, \varepsilon_{(3)}^r$ are three *fixed* vectors. They are fixed if their directions are fixed, since they are unit vectors. As a special coordinate system, then, let us take Cartesian rectangular axes $\bar{x}, \bar{y}, \bar{z}$ with $O\bar{x}, O\bar{y}, O\bar{z}$ in the directions of $d\alpha, d\beta, d\gamma$ at the point concerned. For this system of coordinates $\varepsilon_{(1)}^r, \varepsilon_{(2)}^r, \varepsilon_{(3)}^r$ have components (1, 0, 0), (0, 1, 0), (0, 0, 1) respectively, so that

$$A_{\alpha\alpha} = \bar{A}_{11}, \quad A_{\beta\beta} = \bar{A}_{22}, \quad \text{etc.}$$

Thus the physical components of A_{rs} are simply the components of the tensor referred to $O\bar{x}, O\bar{y}, O\bar{z}$.

For example, if A_{rs} is the strain tensor e_{rs} defined on page 272, then

$$A_{\alpha\alpha} = \frac{\partial \bar{u}}{\partial \bar{x}}, \quad A_{\beta\gamma} = \frac{1}{2} \frac{\partial \bar{v}}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{y}} \quad \text{etc.,}$$

where $\bar{u}, \bar{v}, \bar{w}$ are the components of the displacement vector referred to $O\bar{x}, O\bar{y}, O\bar{z}$.

We can obviously extend the definition of physical components to tensors of any order.

3. Dynamics.

We shall obtain the equations of motion of a particle in the coordinate system (α, β, γ) .

Let $(v_\alpha, v_\beta, v_\gamma)$ be the physical components of the velocity vector. Then, since $v^1 = \dot{\alpha}$, $v^2 = \dot{\beta}$, $v^3 = \dot{\gamma}$, we see from (7) that

$$v_\alpha = \frac{\dot{\alpha}}{h_1}, \quad v_\beta = \frac{\dot{\beta}}{h_2}, \quad v_\gamma = \frac{\dot{\gamma}}{h_3} \quad \dots \quad (9)$$

The acceleration vector may be found by substituting directly in (4), p. 219, or as follows. Using (10), p. 222, we have

$$f_1 = \frac{d}{dt} \left(\frac{\partial \varphi}{\partial \dot{\alpha}} \right) - \frac{\partial \varphi}{\partial \alpha},$$

where

$$\varphi = \frac{1}{2} \left\{ \left(\frac{\dot{\alpha}}{h_1} \right)^2 + \left(\frac{\dot{\beta}}{h_2} \right)^2 + \left(\frac{\dot{\gamma}}{h_3} \right)^2 \right\}.$$

Hence the physical components of the acceleration vector are

$$f_\alpha = h_1 f_1 = h_1 \frac{d}{dt} \left(\frac{\dot{\alpha}}{h_1} \right) + h_1 \left(\frac{\partial \log h_1}{\partial \alpha} \frac{\dot{\alpha}^2}{h_1^2} + \frac{\partial \log h_2}{\partial \alpha} \frac{\dot{\beta}^2}{h_2^2} + \frac{\partial \log h_3}{\partial \alpha} \frac{\dot{\gamma}^2}{h_3^2} \right), \text{ etc.}, \quad (10)$$

the other two components being written down by symmetry.

The physical components of the force vector ($Q_\alpha, Q_\beta, Q_\gamma$) are given by

$$\left. \begin{aligned} Q_\alpha &= h_1 Q_1 = -h_1 \frac{\partial V}{\partial \alpha}, & Q_\beta &= h_2 Q_2 = -h_2 \frac{\partial V}{\partial \beta}, \\ Q_\gamma &= h_3 Q_3 = -h_3 \frac{\partial V}{\partial \gamma}, \end{aligned} \right\} \quad (11)$$

V being the potential function; and the equations of motion are

$$M f_\alpha = Q_\alpha, \quad M f_\beta = Q_\beta, \quad M f_\gamma = Q_\gamma. \quad \dots \quad (12)$$

4. Electricity.

We use the following notation for the physical components of the electromagnetic field-vectors: (i) ($E_\alpha, E_\beta, E_\gamma$) the electric force vector, (ii) ($D_\alpha, D_\beta, D_\gamma$) the displacement vector, (iii) ($H_\alpha, H_\beta, H_\gamma$) the magnetic force vector, (iv) ($B_\alpha, B_\beta, B_\gamma$) the magnetic induction vector, and (v) ($C_\alpha, C_\beta, C_\gamma$) the current vector.

From (35), p. 267, we see that

$$\frac{1}{c} \frac{\partial B_\alpha}{\partial t} = \frac{1}{c} g_{mn} \frac{\partial B^m}{\partial t} \varepsilon_{(1)}^n = \frac{1}{c h_1} \frac{\partial B^1}{\partial t} = h_2 h_3 \left(\frac{\partial E_2}{\partial \gamma} - \frac{\partial E_3}{\partial \beta} \right),$$

that is,

$$\frac{1}{c} \frac{\partial B_\alpha}{\partial t} = h_2 h_3 \left[\frac{\partial}{\partial \gamma} \left(\frac{E_\beta}{h_2} \right) - \frac{\partial}{\partial \beta} \left(\frac{E_\gamma}{h_3} \right) \right], \quad \dots \quad (13)$$

and the other two equations follow from symmetry. Similarly

$$\frac{1}{c} C_{\alpha} = h_2 h_3 \left[\frac{\partial}{\partial \beta} \left(\frac{H_{\gamma}}{h_3} \right) - \frac{\partial}{\partial \gamma} \left(\frac{H_{\beta}}{h_2} \right) \right], \text{ etc. } \quad (14)$$

Also, from (38), p. 267, we have $D^r_{,r} = \varrho$; but (Ex. 5, p. 155)

$$D^r_{,r} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r} (\sqrt{g} D^r).$$

Hence

$$h_1 h_2 h_3 \left[\frac{\partial}{\partial \alpha} \left(\frac{D_{\alpha}}{h_2 h_3} \right) + \frac{\partial}{\partial \beta} \left(\frac{D_{\beta}}{h_3 h_1} \right) + \frac{\partial}{\partial \gamma} \left(\frac{D_{\gamma}}{h_1 h_2} \right) \right] = \varrho, \quad (15)$$

and similarly

$$h_1 h_2 h_3 \left[\frac{\partial}{\partial \alpha} \left(\frac{B_{\alpha}}{h_2 h_3} \right) + \frac{\partial}{\partial \beta} \left(\frac{B_{\beta}}{h_3 h_1} \right) + \frac{\partial}{\partial \gamma} \left(\frac{B_{\gamma}}{h_1 h_2} \right) \right] = 0. \quad (16)$$

Equations (13)–(16) are the electromagnetic equations in our curvilinear coordinate system.

For the electrostatic field in a vacuum there exists a potential function V , which satisfies Poisson's equation $\Delta V = -\varrho$, and this can be written

$$\Delta V = h_1 h_2 h_3 \left\{ \begin{aligned} & \left[\frac{\partial}{\partial \alpha} \left(\frac{h_1}{h_2 h_3} \frac{\partial V}{\partial \alpha} \right) + \frac{\partial}{\partial \beta} \left(\frac{h_2}{h_3 h_1} \frac{\partial V}{\partial \beta} \right) \right. \\ & \left. + \frac{\partial}{\partial \gamma} \left(\frac{h_3}{h_1 h_2} \frac{\partial V}{\partial \gamma} \right) \right] = -\varrho, \end{aligned} \right\} \quad (17)$$

by using Ex. 6, p. 155. It gives the expression for the Laplacian of V in orthogonal curvilinear coordinates.

5. Elasticity.

If we denote by $(u_{\alpha}, u_{\beta}, u_{\gamma})$ the projections of the displacement vector ξ_r of p. 271, we have

$$u_{\alpha} = h_1 \xi_1 = \frac{\xi_1}{h_1}, \quad u_{\beta} = h_2 \xi_2 = \frac{\xi_2}{h_2}, \quad u_{\gamma} = h_3 \xi_3 = \frac{\xi_3}{h_3}.$$

The physical components of the strain tensor e_{rs} are therefore

$$\begin{aligned} e_{\alpha\alpha} &= h_1^2 e_{11} = h_1^2 \xi_{1,1} = h_1^2 \left[\frac{\partial \xi_1}{\partial \alpha} - \left\{ \begin{matrix} p \\ 11 \end{matrix} \right\} \xi_p \right], \text{ etc.,} \\ e_{\beta\gamma} &= h_2 h_3 e_{23} = \frac{1}{2} h_2 h_3 (\xi_{2,3} + \xi_{3,2}) \\ &= \frac{1}{2} h_2 h_3 \left[\frac{\partial \xi_2}{\partial \gamma} + \frac{\partial \xi_3}{\partial \beta} - 2 \left\{ \begin{matrix} p \\ 23 \end{matrix} \right\} \xi_p \right], \text{ etc.} \end{aligned}$$

On substituting the values of the Christoffel symbols from (5), these formulae can be reduced to the form

$$\begin{aligned} e_{\alpha\alpha} &= h_1 \frac{\partial u_\alpha}{\partial \alpha} - h_2 \frac{\partial \log h_1}{\partial \beta} u_\beta - h_3 \frac{\partial \log h_1}{\partial \gamma} u_\gamma, \text{ etc.}, \\ e_{\beta\gamma} &= \frac{1}{2} \left[\frac{h_2}{h_3} \frac{\partial}{\partial \beta} (h_3 u_\gamma) + \frac{h_3}{h_2} \frac{\partial}{\partial \gamma} (h_2 u_\beta) \right], \text{ etc.} \end{aligned} \quad (18)$$

Also the physical components of the rotation tensor ω_{rs} are

$$\begin{aligned} \omega_{\beta\gamma} &= h_2 h_3 \omega_{23} = \frac{1}{2} h_2 h_3 [\xi_{2,3} - \xi_{3,2}] \\ &= \frac{1}{2} h_2 h_3 \left[\frac{\partial \xi_2}{\partial \gamma} - \frac{\partial \xi_3}{\partial \beta} \right], \text{ etc.}, \end{aligned}$$

that is,

$$\omega_{\beta\gamma} = \frac{1}{2} h_2 h_3 \left[\frac{\partial}{\partial \gamma} \left(\frac{u_\beta}{h_2} \right) - \frac{\partial}{\partial \beta} \left(\frac{u_\gamma}{h_3} \right) \right], \text{ etc.} \quad (19)$$

The physical components of the stress tensor E^{rs} are denoted by $\widehat{\alpha\alpha}$, $\widehat{\beta\gamma}$, etc.; thus

$$\alpha\alpha = \frac{E^{11}}{h_1^2}, \quad \widehat{\beta\gamma} = \frac{E^{23}}{h_2 h_3}, \text{ etc.}$$

Now the equations of motion of a continuous medium are given by (10), p. 275. Expressing these in physical components we get

$$\begin{aligned} \varrho (f_\alpha - F_\alpha) &= \frac{1}{h_1} E^{1..m}_{..m} \\ &= \frac{1}{h_1} \left[\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^m} (\sqrt{g} E^{1m}) + \left\{ \frac{1}{m n} \right\} E^{mn} \right], \end{aligned}$$

(cf. Ex. 7, p. 155). We have then

$$\begin{aligned} E^{1..m}_{..m} &= h_1 h_2 h_3 \left[\frac{\partial}{\partial \alpha} \left(\frac{E^{11}}{h_1 h_2 h_3} \right) + \frac{\partial}{\partial \beta} \left(\frac{E^{12}}{h_1 h_2 h_3} \right) + \frac{\partial}{\partial \gamma} \left(\frac{E^{13}}{h_1 h_2 h_3} \right) \right] \\ &\quad - \frac{\partial \log h_1}{\partial \alpha} E^{11} + \frac{h_1^2}{h_2^2} \frac{\partial \log h_2}{\partial \alpha} E^{22} \\ &\quad + \frac{h_1^2}{h_3^2} \frac{\partial \log h_3}{\partial \alpha} E^{33} - \frac{\partial \log h_1}{\partial \beta} E^{12} - \frac{\partial \log h_1}{\partial \gamma} E^{13}. \end{aligned}$$

Introducing the physical components of E^{rs} , we obtain the equations of motion in the final form

$$\left. \begin{aligned} \varrho(f_\alpha - F_\alpha) = & h_1 h_2 h_3 \left[\frac{\partial}{\partial \alpha} \left(\frac{\widehat{\alpha\alpha}}{h_2 h_3} \right) + h_1 \frac{\partial}{\partial \beta} \left(\frac{\widehat{\alpha\beta}}{h_1^2 h_3} \right) \right. \\ & + h_1 \frac{\partial}{\partial \gamma} \left(\frac{\widehat{\alpha\gamma}}{h_1^2 h_2} \right) \Big] \\ & + h_1 \frac{\partial \log h_2}{\partial \alpha} \widehat{\beta\beta} + h_1 \frac{\partial \log h_3}{\partial \alpha} \widehat{\gamma\gamma}, \text{ etc.} \end{aligned} \right\} \quad (20)$$

Also the formula for the dilatation θ is

$$\left. \begin{aligned} \theta = \xi^r_{,r} = & \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r} (\sqrt{g} \xi^r) \\ = & h_1 h_2 h_3 \left[\frac{\partial}{\partial \alpha} \left(\frac{u_\alpha}{h_2 h_3} \right) + \frac{\partial}{\partial \beta} \left(\frac{u_\beta}{h_3 h_1} \right) + \frac{\partial}{\partial \gamma} \left(\frac{u_\gamma}{h_1 h_2} \right) \right]. \end{aligned} \right\} \quad (21)$$

6. Hydrodynamics.

In the motion of a fluid the acceleration vector f_r is given by (15), p. 276. Hence its physical components are

$$\begin{aligned} f_\alpha &= h_1 f_1 = h_1 \left[\frac{\partial v_1}{\partial t} + v_{1,m} v^m \right] \\ &= h_1 \left[\frac{\partial v_1}{\partial t} + (v_{1,m} - v_{m,1}) v^m + \frac{1}{2} \frac{\partial}{\partial x^1} (v_m v^m) \right], \end{aligned}$$

since $\frac{\partial}{\partial x^r} (v_m v^m) = 2v_{m,r} v^m$; that is,

$$\begin{aligned} f_\alpha &= h_1 \left[\frac{\partial}{\partial t} \left(\frac{v_\alpha}{h_1} \right) + \left\{ \frac{\partial}{\partial \beta} \left(\frac{v_\alpha}{h_1} \right) - \frac{\partial}{\partial \alpha} \left(\frac{v_\beta}{h_2} \right) \right\} h_2 v_\beta \right. \\ &\quad \left. + \left\{ \frac{\partial}{\partial \gamma} \left(\frac{v_\alpha}{h_1} \right) - \frac{\partial}{\partial \alpha} \left(\frac{v_\gamma}{h_3} \right) \right\} h_3 v_\gamma + \frac{1}{2} \frac{\partial}{\partial \alpha} (v_\alpha^2 + v_\beta^2 + v_\gamma^2) \right], \end{aligned}$$

where $v_\alpha, v_\beta, v_\gamma$ are the physical components of the velocity vector. Therefore

$$\begin{aligned} f_\alpha &= \frac{\partial v_\alpha}{\partial t} + h_1 \left[v_\alpha \frac{\partial v_\alpha}{\partial \alpha} + h_2 v_\beta \frac{\partial}{\partial \beta} \left(\frac{v_\alpha}{h_1} \right) \right. \\ &\quad \left. + h_3 v_\gamma \frac{\partial}{\partial \gamma} \left(\frac{v_\alpha}{h_1} \right) + \frac{\partial \log h_2}{\partial \alpha} v_\beta^2 + \frac{\partial \log h_3}{\partial \alpha} v_\gamma^2 \right], \text{ etc.} \end{aligned} \quad (22)$$

The equations of motion are derived from (12), p. 276, and are

$$\frac{1}{\varrho} \frac{\partial p}{\partial \alpha} = \frac{1}{h_1} (F_\alpha - f_\alpha), \text{ etc. } \dots \dots (23)$$

Also from (17), p. 277, we see that the equation of continuity can be written

$$\frac{\partial \varrho}{\partial t} + (\varrho v^r)_{,r} = \frac{\partial \varrho}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r} (\sqrt{g} \varrho v^r) = 0,$$

or, in terms of the physical components,

$$\left. \begin{aligned} \frac{\partial \varrho}{\partial t} + h_1 h_2 h_3 \left[\frac{\partial}{\partial \alpha} \left(\frac{\varrho v_\alpha}{h_2 h_3} \right) + \frac{\partial}{\partial \beta} \left(\frac{\varrho v_\beta}{h_3 h_1} \right) \right. \\ \left. + \frac{\partial}{\partial \gamma} \left(\frac{\varrho v_\gamma}{h_1 h_2} \right) \right] = 0. \end{aligned} \right\} \dots (24)$$

EXAMPLES.

1. *Spherical polar coordinates.* Establish the following results for spherical polar coordinates (r, θ, φ) :

(i) *Line-element.*

$$ds^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\varphi)^2.$$

(ii) *Dynamics.*

$$M \{ \ddot{r} - r \dot{\theta}^2 - r \sin^2 \theta \dot{\varphi}^2 \} = Q_r = - \frac{\partial V}{\partial r},$$

$$M \left\{ \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) - r \sin \theta \cos \theta \dot{\varphi}^2 \right\} = Q_\theta = - \frac{1}{r} \frac{\partial V}{\partial \theta},$$

$$M \left\{ \frac{1}{r \sin \theta} \frac{d}{dt} (r^2 \sin^2 \theta \dot{\varphi}) \right\} = Q_\varphi = - \frac{1}{r \sin \theta} \frac{\partial V}{\partial \varphi}.$$

(iii) *Electricity.*

$$\frac{1}{c} \frac{\partial B_r}{\partial t} - \frac{1}{r \sin \theta} \left[\frac{\partial E_\theta}{\partial \varphi} - \frac{\partial}{\partial \theta} (\sin \theta E_\varphi) \right],$$

$$\frac{1}{c} \frac{\partial B_\theta}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} (r E_\varphi) - \frac{1}{r \sin \theta} \frac{\partial E_r}{\partial \varphi},$$

$$\frac{1}{c} \frac{\partial B_\varphi}{\partial t} = \frac{1}{r} \left[\frac{\partial E_r}{\partial \theta} - \frac{\partial}{\partial r} (r E_\theta) \right].$$

$$\frac{1}{c} C_r = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta H_\varphi) - \frac{\partial H_\theta}{\partial \varphi} \right]$$

$$\frac{1}{c} C_\theta = \frac{1}{r \sin \theta} \frac{\partial H_r}{\partial \varphi} - \frac{1}{r} \frac{\partial}{\partial r} (r H_\varphi),$$

$$\frac{1}{c} C_\varphi = \frac{1}{r} \left[\frac{\partial}{\partial r} (r H_\theta) - \frac{\partial H_r}{\partial \theta} \right].$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 D_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta D_\theta) + \frac{1}{r \sin \theta} \frac{\partial D_\varphi}{\partial \varphi} = e,$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 B_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta B_\theta) + \frac{1}{r \sin \theta} \frac{\partial B_\varphi}{\partial \varphi} = 0.$$

$$\Delta V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \varphi^2}.$$

(iv) *Elasticity.*

$$e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r},$$

$$e_{\varphi\varphi} = \frac{1}{r \sin \theta} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} + \frac{\cot \theta}{r} u_\theta,$$

$$e_{\theta\varphi} = \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \varphi} + \frac{1}{r} \frac{\partial u_\varphi}{\partial \theta} - \frac{\cot \theta}{r} u_\varphi \right),$$

$$e_{r\varphi} = \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r} \right),$$

$$e_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right).$$

$$\omega_{\theta\varphi} = \frac{1}{2r \sin \theta} \left[\frac{\partial u_\theta}{\partial \varphi} - \frac{\partial}{\partial \theta} (\sin \theta u_\varphi) \right],$$

$$\omega_{r\varphi} = \frac{1}{2} \left[\frac{1}{r} \frac{\partial}{\partial r} (r u_\varphi) - \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \varphi} \right],$$

$$\omega_{r\theta} = \frac{1}{2r} \left[\frac{\partial u_r}{\partial \theta} - \frac{\partial}{\partial r} (r u_\theta) \right].$$

$$\begin{aligned} e(f_r - F_r) &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \widehat{r r}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \widehat{r \theta}) \\ &\quad + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (r \widehat{\varphi}) - \frac{1}{r} (\widehat{\theta \theta} + \widehat{\varphi \varphi}), \end{aligned}$$

$$\begin{aligned} e(f_\theta - F_\theta) &= \frac{1}{r^3} \frac{\partial}{\partial r} (r^3 \widehat{\theta r}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \widehat{\theta \theta}) \\ &\quad + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\widehat{\theta \varphi}) - \frac{\cot \theta}{r} \widehat{\varphi \varphi}, \end{aligned}$$

$$\begin{aligned} e(f_\varphi - F_\varphi) &= \frac{1}{r^3} \frac{\partial}{\partial r} (r^3 \widehat{\varphi r}) + \frac{1}{r \sin^2 \theta} \frac{\partial}{\partial \theta} (\sin^2 \theta \widehat{\varphi \theta}) \\ &\quad + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\widehat{\varphi \varphi}). \end{aligned}$$

(v) *Hydrodynamics.*

$$\begin{aligned}
 f_r &= \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\varphi}{r \sin \theta} \frac{\partial v_r}{\partial \varphi} - \frac{1}{r} (v_\theta^2 + v_\varphi^2), \\
 f_\theta &= \frac{\partial v_\theta}{\partial t} + \frac{v_r}{r} \frac{\partial}{\partial r} (r v_\theta) + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\varphi}{r \sin \theta} \frac{\partial v_\theta}{\partial \varphi} - \frac{\cot \theta}{r} v_\varphi^2, \\
 f_\varphi &= \frac{\partial v_\varphi}{\partial t} + \frac{v_r}{r} \frac{\partial}{\partial r} (r v_\varphi) + \frac{v_\theta}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\varphi) + \frac{v_\varphi}{r \sin \theta} \frac{\partial v_\varphi}{\partial \varphi}. \\
 \frac{1}{\varrho} \frac{\partial p}{\partial r} &= (F_r - f_r), \quad \frac{1}{\varrho} \frac{\partial p}{\partial \theta} = r (F_\theta - f_\theta), \\
 \frac{1}{\varrho} \frac{\partial p}{\partial \varphi} &= r \sin \theta (F_\varphi - f_\varphi).
 \end{aligned}$$

$$\frac{\partial \varrho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \varrho v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \varrho v_\theta) + \frac{1}{r \sin \theta} \frac{\partial (\varrho v_\varphi)}{\partial \varphi} = 0.$$

2. *Cylindrical coordinates.* Establish the following results for cylindrical coordinates (r, θ, z) :

(i) *Line-element.*

$$ds^2 = (dr)^2 + r^2 (d\theta)^2 + (dz)^2.$$

(ii) *Dynamics.*

$$\begin{aligned}
 M \{ \ddot{r} - r \dot{\theta}^2 \} &= Q_r = - \frac{\partial V}{\partial r}, \quad M \left\{ \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) \right\} = Q_\theta = - \frac{1}{r} \frac{\partial V}{\partial \theta}, \\
 M \ddot{z} &= Q_z = - \frac{\partial V}{\partial z}.
 \end{aligned}$$

(iii) *Electricity.*

$$\begin{aligned}
 \frac{1}{c} \frac{\partial B_r}{\partial t} &= \frac{\partial E_\theta}{\partial z} - \frac{1}{r} \frac{\partial E_z}{\partial \theta}, \quad \frac{1}{c} \frac{\partial B_\theta}{\partial t} = \frac{\partial E_z}{\partial r} - \frac{\partial E_r}{\partial z}, \\
 \frac{1}{c} \frac{\partial B_z}{\partial t} &= \frac{1}{r} \left[\frac{\partial E_r}{\partial \theta} - \frac{\partial}{\partial r} (r E_\theta) \right], \\
 \frac{1}{c} C_r &= \frac{1}{r} \frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z}, \quad \frac{1}{c} C_\theta = \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}, \\
 \frac{1}{c} C_z &= \frac{1}{r} \left[\frac{\partial}{\partial r} (r H_\theta) - \frac{\partial H_r}{\partial \theta} \right], \\
 \frac{1}{r} \frac{\partial}{\partial r} (r D_r) &+ \frac{1}{r} \frac{\partial D_\theta}{\partial \theta} + \frac{\partial D_z}{\partial z} = \varrho, \\
 \frac{1}{r} \frac{\partial}{\partial r} (r B_r) &+ \frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0, \\
 \Delta V &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}.
 \end{aligned}$$

(iv) *Elasticity.*

$$\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r}, & e_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, & e_{zz} &= \frac{\partial u_z}{\partial z}, \\
e_{\theta z} &= \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right), & e_{zr} &= \frac{1}{2} \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right), \\
e_{r\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \\
\omega_{\theta z} &= \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} - \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right), & \omega_{zr} &= \frac{1}{2} \left(\frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z} \right), \\
\omega_{r\theta} &= \frac{1}{2r} \left[\frac{\partial u_r}{\partial \theta} - \frac{\partial}{\partial r} (r u_\theta) \right], \\
\varrho (f_r - F_r) &= \frac{1}{r} \frac{\partial}{\partial r} (r \widehat{rr}) + \frac{1}{r} \frac{\partial}{\partial \theta} (r \widehat{\theta}) + \frac{\partial}{\partial z} (r \widehat{z}) - \frac{\widehat{\theta}\theta}{r}, \\
\varrho (f_\theta - F_\theta) &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \widehat{\theta r}) + \frac{1}{r} \frac{\partial}{\partial \theta} (\widehat{\theta\theta}) + \frac{\partial}{\partial z} (\widehat{\theta z}), \\
\varrho (f_z - F_z) &= \frac{1}{r} \frac{\partial}{\partial r} (r \widehat{zr}) + \frac{1}{r} \frac{\partial}{\partial \theta} (\widehat{z\theta}) + \frac{\partial}{\partial z} (\widehat{zz}).
\end{aligned}$$

(v) *Hydrodynamics.*

$$\begin{aligned}
f_r &= \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}, \\
f_\theta &= \frac{\partial v_\theta}{\partial t} + \frac{v_r}{r} \frac{\partial}{\partial r} (r v_\theta) + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z}, \\
f_z &= \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}, \\
\frac{1}{\varrho} \frac{\partial p}{\partial r} &= (F_r - f_r), & \frac{1}{\varrho} \frac{\partial p}{\partial \theta} &= r (F_\theta - f_\theta), & \frac{1}{\varrho} \frac{\partial p}{\partial z} &= F_z - f_z, \\
\frac{\partial \varrho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \varrho v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\varrho v_\theta) + \frac{\partial}{\partial z} (\varrho v_z) &= 0.
\end{aligned}$$

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